A Pedagogic Note on the Derivation of the Black-Scholes Option Pricing Formula

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1. Introduction

Although the Black and Scholes (1973) option pricing formula is widely used by the finance profession in numerous research and teaching applications, its underlying economic and mathematical structure is generally not well known. The purposes of this note are the following: (1) to elucidate the economic and mathematical arguments that are needed to derive the Black-Scholes formula and (2) to demonstrate an alternative derivation based upon the risk neutrality arguments of Cox and Ross (1976) and Rubinstein (1976). Although Borch (1984) has previously derived the Black-Scholes formula using a similar mathematical framework, we hope that both the economics and mathematics underlying our derivation are more transparent.

The note is organized as follows. In the next section, we discuss the economic assumptions that underlie the pricing of options in both continuous and discrete time. Since our focus is on the derivation of the Black-Scholes option pricing formula per se, we limit our discussion to the assumptions that are needed for the derivation of that particular formula. In the third section, we summarize some important properties of normal and lognormal variables, which will prove to be quite useful in our derivation. The note concludes with a step-by-step derivation of the Black-Scholes option pricing formula.

2. Economic Structure of the Black-Scholes Model

Perhaps the most important economic insight contained in the seminal paper by Cox and Ross is the notion that investors can create essentially riskless securities by constructing portfolios consisting of call options and their underlying assets in proportions such that instantaneous portfolio returns are nonstochastic. Mathematically, this riskless hedging argument implies a partial differential equation relating the value of the call option to the value of the underlying asset, which can be solved by transforming it into the heat transfer equation from physics.

Cox and Ross suggest a more intuitive solution technique that has become known in the literature as the risk neutrality argument. Since the creation of a riskless hedge places no restrictions on investor preferences beyond nonsatiation, the valuation relationship between

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the option and its underlying asset is independent of investor risk preferences. That is, for a given price of the underlying asset, a call option written against that asset will trade at the same price in a risk neutral economy as it will in a risk averse or risk preferent economy. This rather convenient result enables us to price options as if they are traded in a risk neutral economy without any loss of generality.

If security trading is allowed to take place only at discrete intervals, then the riskless hedging argument of Black and Scholes will generally break down. However, Rubinstein has shown that the Black-Scholes formula obtains in discrete time if the following assumptions are made:

1. The conditions for aggregation are met so that securities are priced as if all investors have the same characteristics as a representative investor.
2. The utility function of the representative investor exhibits constant proportional risk aversion.
3. The return on the underlying asset and the return on aggregate wealth are bivariate lognormally distributed.
4. The return on the underlying asset follows a stationary random walk through time.
5. The riskless rate of interest is constant through time.

Collectively, assumptions 1, 2, and 3 constitute sufficient conditions for valuing the option as if both it and the underlying asset are traded in a risk neutral economy, even though risk aversion in fact plays a key role in the determination of the equilibrium value of the underlying asset. However, in order for the discrete time formula to be identical to the Black-Scholes formula, assumptions 4 and 5 are also needed. These latter two assumptions constitute sufficient conditions for the rate of return on the underlying asset, its variance rate, and the riskless rate of interest to be time homogeneous; that is, proportional to time.

3. Important Properties of Normal and Lognormal Variables

Before we derive the Black-Scholes formula, a brief discussion of some important properties of normal and lognormal variables and their distribution functions, density functions, and central moments is in order. Suppose there exists a random variable x that is normally distributed with mean μ_x and variance σ_x^2. The density function of x, $f(x)$, is given by equation (1):

$$f(x) = \left(2\pi\sigma_x^2\right)^{-.5} e^{-\{.5[(x-\mu_x)^2/\sigma_x^2]\}} \tag{1}$$

Next, we define a lognormal random variable $y = e^x$. The distribution functions of x and y are related to each other in the following manner:

$$F(x) = G(y), \tag{2}$$

where

$F(x) = \int_{-\infty}^{x} f(x)dx;$
$G(y) = \int_{0}^{y} g(y)dy;$ and
$g(y) =$ the density function of y.

Since the density function $g(y)$ is equal to $dG(y)/dy$, the relationship between the normal density function $f(x)$ and the lognormal density function $g(y)$ is easily seen by differentiating $F(x)$ with respect to y:

$$\begin{aligned} g(y) &= dF(x)/dy \\ &= (dF(x)/dx)(dx/dy) \\ &= f(x)(1/y). \end{aligned} \tag{3}$$

Thus, the density function of a lognormal random variable is proportional to the density of its natural logarithm, the factor of proportionality in this case being the lognormal variable's reciprocal. Finally, the mean and variance of y, μ_y and σ_y^2, are related to the mean and variance of x by

$$\mu_y = e^{(\mu_x + .5\sigma_x^2)}, \text{ and} \tag{4}$$

$$\sigma_y^2 = e^{(2\mu_x + \sigma_x^2)}(e^{\sigma_x^2} - 1). \tag{5}$$

4. Derivation of the Black-Scholes Formula

In view of the risk neutrality arguments of Cox and Ross and Rubinstein, today's option price can be determined by discounting the expected value of the terminal option price in a risk neutral economy by the riskless rate of interest. The present value of the expected terminal option price is equal to the present value of the difference between the expected terminal stock price and the exercise price, conditional upon the call option expiring in-the-money; that is,

$$\begin{aligned} C &= e^{-rt}E[Max(S_t - X, 0)] \\ &= e^{-rt}\int_{X}^{\infty}(S_t - X)h(S_t)dS_t, \end{aligned} \tag{6}$$

where

$C =$ the market value of the call option;
$r =$ the riskless rate of interest;
$t =$ the time to expiration;
$S_t =$ the market value of the underlying stock at time t;
$X =$ the exercise or striking price; and

$h(S_t)$ = the "risk neutral" lognormal density function of S_t.[1]

To evaluate this integral, we rewrite it as the difference between two integrals:

$$\begin{aligned} C &= e^{-rt}\left[\int_X^\infty S_t h(S_t)dS_t - X\int_X^\infty h(S_t)dS_t\right] \\ &= E_X(S_t)e^{-rt} - Xe^{-rt}[1-H(X)], \end{aligned} \tag{7}$$

where

$E_X(S_t)$ = the partial expectation of St, truncated from below at X; and
$H(X)$ = the probability that $S_t \leq X$.

By rewriting the terminal stock price S_t as the product of the current stock price S and the t-period lognormally distributed price ratio S_t/S, $S_t = S(S_t/S)$, equation (7) can be rewritten as

$$\begin{aligned} C &= e^{-rt}\left[S\int_{X/S}^\infty (S_t/S)g(S_t/S)dS_t/S - X\int_{X/S}^\infty g(S_t/S)dS_t/S\right] \\ &= Se^{-rt}E_{X/S}(S_t/S) - Xe^{-rt}[1-G(X/S)], \end{aligned} \tag{8}$$

where

$g(S_t/S)$ = lognormal density function of S_t/S;
$E_{X/S}(S_t/S)$ = the partial expectation of S_t/S, truncated from below at X/S;
$G(X/S)$ = the probability that $S_t/S \leq X/S$;

Next, we evaluate the right-hand side of equation (8) by considering its two integrals separately. First, consider the integral that corresponds to the present value of the partial expectation of the terminal stock price, $Se^{-rt}E_{X/S}(S_t/S)$. By assuming that the return on the underlying asset follows a stationary random walk, we are able to rewrite the price ratio S_t/S as a function of time; that is, $S_t/S = e^{kt}$, where k is the rate of return on the underlying asset per unit time. Taking the natural logarithm of both sides of this expression, we find that $\ln(S_t/S) = kt$. Since the ratio S_t/S is lognormally distributed, it follows that kt is normally distributed with density $f(kt)$, mean $\mu_k t$ and variance $\sigma_k^2 t$. Furthermore, from equation (3), we know that $g(S_t/S) = (S_t/S)f(kt)$. Also, since $S_t/S = e^{kt}$, we can write the differential $dS_t/S = e^{kt}tdk$. Therefore, our logarithmic transformation $ln(S_t/S) = kt$ enables us to rewrite the product $(S_t/S)g(S_t/S)dS_t/S$ in equation (7) as $f(kt)e^{kt}tdk$. Hence,

$$\begin{aligned} Se^{-rt}E_{X/S}(S_t/S) &= Se^{-rt}\int_{\ln(X/S)}^\infty f(kt)e^{-kt}tdk \\ &= Se^{-rt}\left(2\pi\sigma_k^2 t\right)^{-.5}\int_{\ln(X/S)}^\infty e^{kt}e^{-\{.5[(kt-\mu_k t)^2/\sigma_k^2 t]\}}tdk. \end{aligned} \tag{9}$$

[1] The $h(S_T)$ density function is "risk neutral" in the sense that its location parameter μt is replaced by rt. Here, μ corresponds to the (annualized) expected return on the underlying asset, whereas r corresponds to the (annualized) riskless rate of interest.

Next, we simplify equation (9)'s integrand by adding the terms in the two exponents, multiplying and dividing the resulting expression by $e^{-.5\sigma_k^2 t}$, and rearranging:

$$\begin{aligned}
e^{kt}e^{-\{.5[(kt-\mu_k t)^2/\sigma_k^2 t]\}} &= e^{-\{.5t[(k^2-2\mu_k k+\mu_k^2-2\sigma_k^2 k)/\sigma_k^2]\}} \\
&= e^{-\{.5t[(k^2-2\mu_k k+\mu_k^2-2\sigma_k^2 k+\sigma_k^4-\sigma_k^4)/\sigma_k^2]\}} \\
&= e^{-\{.5t[((k-\mu_k-\sigma_k^2)^2-\sigma_k^4-2\mu_k\sigma_k^2)/\sigma_k^2]\}} \\
&= e^{(\mu_k+.5\sigma_k^2)t}e^{-\{.5[(kt-(\mu_k+\sigma_k^2)t)^2/\sigma_k^2 t]\}}. \qquad (10)
\end{aligned}$$

In equation (10), the term $e^{(\mu_k+.5\sigma_k^2)t} = E(S_t/S)$, the mean of the t-period lognormally distributed price ratio S_t/S (cf. equation (4)). Since $E(S_t/S)$ is a constant, we rewrite equation (9) with this term appearing outside the integral:

$$Se^{-rt}E_{X/S}(S_t/S) = SE(S_t/S)e^{-rt}\left(2\pi\sigma_k^2 t\right)^{-.5}\int_{\ln(X/S)}^{\infty} e^{-\{.5[(kt-(\mu_k+\sigma_k^2)t)^2/\sigma_k^2 t]\}}tdk. \qquad (11)$$

Since the equilibrium rate of return in a risk neutral economy is the riskless rate of interest, $E(S_t/S)$ may be rewritten as e^{rt}. Therefore, $SE(S_t/S)e^{-rt} = Se^{rt}e^{-rt} = S$. Hence we rewrite equation (11) as

$$Se^{-rt}E_{X/S}(S_t/S) = S\left(2\pi\sigma_k^2 t\right)^{-.5}\int_{\ln(X/S)}^{\infty} e^{-\{.5[(kt-(\mu_k+\sigma_k^2)t)^2/\sigma_k^2 t]\}}tdk. \qquad (12)$$

To complete our evaluation of the present value of the partial expectation of the terminal stock price, we define a standard normal random variable $y = [kt - (\mu_k + \sigma_k^2)t]/\sigma_k t^{.5}$; hence $kt = (\mu_k + \sigma_k^2)t + \sigma_k t^{.5}y, tdk = \sigma_k t^{.5}dy$, and the lower limit of integration is rewritten as $[\ln(X/S) - (\mu_k + \sigma_k^2)t]/\sigma_k t^{.5}$. In writing equation (11), we made use of the fact that the equilibrium rate of return in a risk neutral economy is the riskless rate of interest, which implied that $e^{(\mu_k+.5\sigma_k^2)t} = e^{rt}$. Taking the natural logarithm of both sides of this expression, we find that $(\mu_k + .5\sigma_k^2)t = rt$. Hence, $(\mu_k + \sigma_k^2)t = (r + .5\sigma_k^2)t$, which enables us to rewrite the lower limit of integration as $-[\ln(X/S) - (r + .5\sigma_k^2)t]/\sigma_k t^{.5} = -d_1$. Substituting these results into equation (12) yields equation (13):

$$Se^{-rt}E_{X/S}(S_t/S) = S\int_{-d_1}^{\infty}[e^{-.5y^2}/(2\pi)^{.5}]dy. \qquad (13)$$

Since y is a standard normal random variable with mean zero, equation (12) can be rewritten as

$$\begin{aligned}
Se^{-rt}E_{X/S}(S_t/S) &= S\int_{-\infty}^{d_1}[e^{-.5y^2}/(2\pi)^{.5}]dy \\
&= SN(d_1), \qquad (14)
\end{aligned}$$

where $N(d_1)$ is the standard normal distribution function evaluated at $y = d_1$.

All that remains in completing the derivation of the Black-Scholes option pricing formula is to evaluate the integral that corresponds to the term $Xe^{-rt}[1 - G(X/S)]$ from equation (8). By making the logarithmic transformation $ln(S_t/S) = kt$, we can rewrite the density

$g(S_t/S)$ as $(S/S_t)f(kt)$ and the differential $dS_t/S = e^{kt}tdk$. Since $S_t/S = e^{kt}$, the product $g(S_t/S)dS_t/S$ can be rewritten $f(kt)tdk$. Hence,

$$\begin{aligned} Xe^{-rt}[1 - G(X/S)] &= Xe^{-rt}\int_{\ln(X/S)}^{\infty} f(kt)tdk \\ &= Xe^{-rt}(2\pi\sigma_k^2 t)^{-.5}\int_{\ln(X/S)}^{\infty} e^{-\{.5[(kt-\mu_k t)^2/\sigma_k^2 t]\}}tdk. \end{aligned} \tag{15}$$

Next, we define a standard normal random variable $z = [kt - \mu_k t]/\sigma_k t^{.5}$; hence $kt = \mu_k t + \sigma_k t^{.5}z, tdk = \sigma_k t^{.5}dz$, and the lower limit of integration is rewritten as $[\ln(X/S) - \mu_k t]/\sigma_k t^{.5}$. Since $e^{(\mu_k + .5\sigma_k^2)t} = e^{rt}$ in a risk neutral economy, we can write $(\mu_k + .5\sigma_k^2)t = rt$, which implies that $\mu_k t = (r - .5\sigma_k^2)t$. Hence, the lower limit of integration can be rewritten as $-[ln(X/S) + (r - .5\sigma_k^2)t]/\sigma_k t^{.5} = -(d_1 - \sigma_k t^{.5}) = -d_2$. Substituting these results into equation (15) yields equation (16):

$$\begin{aligned} Xe^{-rt}[1 - G(X/S)] &= Xe^{-rt}\int_{-d_2}^{\infty}[e^{-.5y^2}/(2\pi)^{.5}]dz \\ &= Xe^{-rt}\int_{-\infty}^{d_2}[e^{-.5y^2}/(2\pi)^{.5}]dz \\ &= Xe^{-rt}N(d_2). \end{aligned} \tag{16}$$

Finally, by substituting the right-hand sides of equations (14) and (16) into equation (8), we obtain the Black-Scholes option pricing formula:

$$C = SN(d_1) - Xe^{-rt}N(d_2). \tag{17}$$

5. References

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