The Geneva Papers on Risk and Insurance Theory, 23: 127–137 (1998)

A Reexamination of the Relationship Between Preferences and Moment Orderings by Rational Risk-Averse Investors

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Abstract

This article examines the relationship between risk, return, skewness, and utility-based preferences. Examples are constructed showing that, for any commonly used utility function, it is possible to have two continuous unimodal random variables X and Y with positive and equal means, X having a larger variance and lower positive skewness than Y, and yet X has larger expected utility than Y, contrary to persistent folklore concerning $U''' > 0$ implying skewness preference for risk averters. In additon, it is shown that *ceteris paribus* analysis of preferences and moments, as occasionally used in the literature, is impossible since equality of higher-order central moments implies the total equality of the distributions involved.

Key words: asset preferences, utility functions, moment orderings, Von Neumann-Morgenstern rationality

1. Introduction

Starting with the seminal work of von Neumann and Morgenstern [1994], a typical approach in modeling the relationship between characteristics of the risks being considered and the attitude toward those risks exhibited by the decision maker has been to postulate the existence of a decision maker's utility function $U(X)$ that reflects preference relationships. Rational economic agents then select from a group of random prospects in order to maximize their expected utility. In this context, the mathematical properties of the utility function $U(X)$ are important for developing stochastic dominance results and are subsequently important for the valuation of the distributions associated with alternative investment opportunities (see Bawa [1982]).

According to Pratt [1964] and Arrow [1971], a reasonable restriction set for the class of investors' utility functions includes $U'(x) > 0$ (nonsatiation), $U''(x) < 0$ (risk aversion or a decreasing marginal utility for wealth), and decreasing absolute risk aversion. In a number of articles (e.g., Arditti [1967]; Booth and Smith [1987]; Kraus and Litzenberger [1976]; and Tsiang [1972], a negative second derivative is actually interpreted as implying an aversion to variance. A positive third derivative is also frequently interpreted as being indicative of skewness preference—that is, preferring a higher third moment for two random variables having equal means and variances (cf. Beedles [1979], p. 72; Jean [1971]; Kraus and Litzenberger [1976], p. 1087; Scott and Horvath [1980], p. 917; and Tsiang [1972], p. 359).

A number of articles (e.g., see Menezes, Geiss, and Tressler [1980]; Meyers [1987]; Rothschild and Stiglitz [1970]; and Whitmore [1970]) have previously addressed the fact that moment preference does not match up with a sequence of utility derivatives. However, some of these old beliefs nevertheless continue to persist. In this article, we systematically gather the evidence showing how and why moment ordering is a necessary but not sufficient condition for stochastic dominance (utility ordering). We prove and show with simple examples that expected utility preferences *never* universally translate into moment preferences—that is, we show that for any arbitrary utility function $U(X)$ possessing the properties $U'(x) > 0$, $U''(x) < 0$, $U'''(x) > 0$, and $U^{(4)}(x) < 0$,[1] there always exist two continuous unimodal random variables X and Y with identical means and variances μ and σ^2 and $E(X-\mu)^3 > E(Y-\mu)^3$ whose preference relationship for the decision maker is exactly reversed—that is, $EU(Y) > EU(X)$.

Justification of skewness preference relating to $U''' > 0$ is often characterized by arguments invoking a *ceteris paribus* condition, which is intended to separate out the effect of higher-order moments (and thus to focus solely on the utility differences resulting from changes in skewness alone or in variance alone). In Section 3, we demonstrate that such *ceteris paribus* conditions are logically impossible since they are, in fact, incompatible with having differing lower-order moments. Specifically, we do this by showing that essentially *any* two random variables with equal higher-order central moments must, in fact, be *identical in distribution*.

When authors attempt to provide a proof to justify their connection between utility preference and moment preference, the following heuristic approach is often one of the routes taken:[2] write a Taylor series expansion for $U(X)$, truncate the series after a few (say, three) terms, and then take expected values. If the first two moments agree for two choice alternatives, and if $U'''(x) > 0$, then expected utility ordering is equivalent to skewness ordering. Although the logical holes in this approach are well known, the magnitude of this problem in the context of commonly used models has not been previously described. In Section 4 we show that, for a distribution and utility pair commonly postulated in economics research, the truncated part that is thrown away can actually be entirely unrelated to the true value when using this truncated Taylor series approximation (and hence, can be even larger in magnitude than the part kept). Section 5 examines the empirical literature supporting the implied connection between moment preference and investor choice. This section shows that the empirical foundations of these analyses are theoretically flawed. Finally, Section 6 presents the conclusions of our analysis.

2. $U'' < 0$ is not aversion to variance, and $U'''(x) > 0$ does not imply skewness preference[3]

A mistaken belief that still persists in some circles is that risk-averse investors prefer greater positive skewness when choosing between alternatives having identical means and variances or that a risk-averse investor will prefer smaller variance when comparing alternatives having equal means (cf. Grauer [1985], p. 1392; Kraus and Litzenberger [1976], p. 1087). This is an incorrect belief because one can always construct two distributions with a given moment ordering for which neither stochastically dominates the other at any degree of stochastic dominance. A simple discrete example illustrates this point.

Example 2.1

Consider the discrete variables X and Y defined by

$$X = \begin{cases} 3.5858 & \text{with probability } .5 \\ 6.4142 & \text{with probability } .5 \end{cases} \quad Y = \begin{cases} 0 & \text{with probability } .02 \\ 5 & \text{with probability } .96 \\ 10 & \text{with probability } .02 \end{cases}$$

Both X and Y are *symmetrically distributed* with mean 5, and all odd order central moments of order 3 or higher set equal to zero. The variance of X is 2 while the variance of Y is 1. The mean-variance decision rule indicates that Y should be considered as only half as risky as X and, because it has the same mean, should be preferred to X using any common utility functions such as $U_1(X) = 1 - \exp\{-X\}$, $U_2(X) = \ln[1 + 2X]$ or $U_3(X) = X^{1/2}$. However, from an expected utility perspective, one can easily calculate that the *reverse* is true and X is preferred to Y for each of the above utility functions.

This is not because of the discreteness of the distributions in the above example. Indeed, as shown below, it is possible to have continuous unimodal distributions for X and Y with equal mean, mode, skewness, and kurtosis but for which variance aversion does not imply risk aversion. Example 2.2 illustrates this.

Example 2.2

Consider the random variables X and Y defined by the density functions:

$$f_X(x) = \begin{cases} 0 & x < 0 \\ .005924433 & 0 \le x < 3.7304 \\ .07827427 & 3.7304 \le x < 10 \\ .069097696 & 10 \le x < 17.05483 \\ 0 & x \ge 17.05483 \end{cases}$$

and

$$f_Y(y) = \begin{cases} 0 & y < 2.9345 \\ .070341938 & 2.9345 \le y < 10 \\ .07241876 & 10 \le y < 16.86646 \\ .001842 & 16.86646 \le y < 20 \\ 0 & y \ge 20. \end{cases}$$

Both X and Y are continuous unimodal distributions supported over the interval [0, 20], with a mean and mode of 10, a skewness of zero and to have equal fourth order central moments of 500. Since the variance of X is 16 while the variance of Y is 16.5, a mean-variance (or a mean-variance-skewness) decision rule would indicate that X would be considered less risky and therefore be strictly preferred to Y. Calculation with the utility $U(X) = 1000(.01 - e^{-X})$ yields $EU(X) = 2.3407 < EU(Y) = 6.2608$, thereby indicating that the *higher variance* variable is preferred from an expected utility standpoint.

The next example (Example 2.3) demonstrates that even if an investor's utility function possesses all of the desirable properties commonly assumed to lead to parsimonious choice (such as risk aversion, decreasing absolute risk aversion, and so on), situations will still exist in which the decision maker will prefer a random prospect possessing simultaneously a *lower* mean, a *higher* variance, and a *lower* skewness. The set of utility functions for which this result holds includes all the common utility functions such as the exponential, mixtures of exponential, logarithmic, power, and so on (see Brockett and Golden [1987]). Similar examples can be constructed for any of these common utility functions, showing that the counterexample is not due to any anomalous utility behavior but is, in fact, a property of all utility functions typically used in rational choice models (except of course, for the quadratic function).

Example 2.3

Consider the continuous, unimodal positive random variables X and Y defined by their piecewise constant density functions, where X has the density function f_x given in Example 2.2, and Y has the density function:

$$f_Y(y) = \begin{cases} 0 & y < 2.42 \\ .070 & 2.42 \leq y < 10 \\ .073 & 10 \leq y < 16.35 \\ .001 & 16.35 \leq y < 20 \\ 0 & y \geq 20. \end{cases}$$

For this pair, X has a higher mean, lower variance, higher skewness, and lower kurtosis than Y, and yet for the common CARA utility function $U(X) = 1000(.1 - \exp\{-X\})$, we have Y preferred to X (a similar example could be constructed for any other common utility function).

3. The impossibility of *ceteris paribus* moment preference analysis

In an attempt to isolate the effect of variance (or skewness) on investors' preferences, authors often assume not only that the first moments of the variables X and Y are as proscribed but in addition, by appending the words *ceteris paribus*, that the fourth- and higher-order central moments agree as well (for example, they might state $\sigma^2(X) < \sigma^2(Y)$ or $\rho(X) > \rho(Y)$, *ceteris paribus*). For example, in explaining the implications of their results, Scott and Horvath ([1980], p. 917) claim that "the investor would be willing to accept a lower expected value from his investment of w in portfolio A, than in portfolio B if both portfolios have the same variance, $\mu_2^A = \mu_2^B$, portfolio A has greater positive skewness $\mu_3^A = \mu_3^B$ and all higher moments are the same." Similarly, Tsiang ([1972], p. 359) states that "a positive skewness of the distribution is a desirable feature, and all other things being equal, a greater skewness would increase the expected utility."

In this section, we demonstrate that *ceteris paribus* constraints on the moment relations creates an internal contradiction, since if the means and higher-order moments are equal, then the two random variables are *identical* in distribution (hence they could not possibly satisfy the condition $\mu_3 > \mu_3$ or have unequal expected utilities).

We state our results for bounded random variables for simplicity (it is not important that the exact bounds be known).[4] The important fact is that the distributions are uniquely determined by their moments.

The key to our proof is the following Lemma (cf. Feller [1968], Lemma 4).

Lemma: *If ν is a measure on $[0, \infty)$, then the Laplace transform of ν*

$$f(t) = \int_0^\infty e^{-tx}\nu(dx),$$

is uniquely determined by its values $\{f(t_j), j = 1, 2, \ldots,\}$ at any set of points $\{t_j\}$ with $\sum_{j=1}^\infty t_j^{-1} = \infty$.

Theorem 1: *Suppose that X and Y are two bounded variables with equal means and with equal central moments above some arbitrarily pre-specified integer $k_0 \geq 2$. Then X and Y have identical distributions.*

Proof. For notational purposes pick an arbitrary common value a and b for which $a \leq X \leq b$ and $a \leq Y \leq b$, and assume that all moments of X and Y of order higher than some $k_0 \geq 2$ are equal.
Let N denote the first odd integer past k_0, and define the random variables

$$Z = \frac{(X-\mu)^N - (a-\mu)^N}{(b-\mu)^N - (a-\mu)^N} \quad \text{and} \quad W = \frac{(Y-\mu)^N - (a-\mu)^N}{(b-\mu)^N - (a-\mu)^N}.$$

Since Z and W are bounded between 0 and 1, we can write $Z = \exp\{-V_1\}$ and $W = \exp\{-V_2\}$ for positive random variables V_1 and V_2. Accordingly, the tth moment of Z and W correspond to the Laplace transform of V_1 and V_2, respectively, evaluated at the point t. Thus, since $EZ^J = EW^j$, we have $f_{v_1}(j) = f_{v_2}(j)$, $j = 1, 2, \ldots,$ where f denotes the Laplace transform. Because $\sum_{j=1}^\infty j^{-1} = \infty$, the entire Laplace transforms of V_1 and V_2 must be equal according to the Lemma. However, since the Laplace transformation exists and uniquely determines the distribution, the distributions of V_1 and V_2 must also be equal. Accordingly, the same must also be true for $(X-\mu)^N$ and $(Y-\mu)^N$, and on taking Nth roots, we see that X and Y must also have identical distributions.

Thus, one cannot isolate variance or skewness preference for decision-makers by the ordering of moments. This shows that while moment ordering is a necessary condition for utility function ordering (stochastic dominance), it is never sufficient. □

4. A reexamination of the approach to justifying moment use by truncating the Taylor series

When developing theoretical results concerning variance or skewness preference and investor choice, a common approach is to write a Taylor series expansion for $U(X)$, truncate the series after two or three terms and then proceed with the analysis as if the utility function were a polynomial utility (so that expected utility can be expressed in terms of moments alone). Aside from the well-known and obvious drawbacks to using quadratic, cubic, or more general polynomial utility functions (for example, the quadratic utility is limited in its range of applicability as a utility, and within that range, the quadratic utility implies *increasing* absolute risk aversion), there are consequential effects of using these truncated polynomial utilities that are economically difficult to justify. Ekern and Wilson [1974] show that all shareholders in this setting prefer a firm characterized by constant stochastic returns that operate so that their market value is zero. Additionally, Borch [1969] has shown that, if preferences satisfy a monotonicity condition, then all indifference curves in the mean-variance plane reduce to a single point with the quadratic utility. For higher-order polynomial utilities, Levy [1969] has shown that a cubic utility function does not exhibit decreasing marginal utility for all wealth levels. In fact, when the utility function has decreasing marginal utility for positive wealth levels near zero, it has increasing absolute risk aversion within that same range. According to Tsiang [1972], polynomial utilities cannot simultaneously satisfy the essential properties of $U' > 0, U'' < 0$ and nonincreasing absolute and proportional risk aversion. At best, this truncation procedure must be considered to provide only an approximation. The question then arises as to how good an approximation is provided by this truncation method.

The next example addresses how far wrong one can go by using a truncated utility capable of reducing preference ordering to moment ordering. It shows that for a frequently used utility function and for a very popular distribution used in modeling,[5] terms can be thrown away that are far larger in magnitude than those terms that have been kept. Moreover, the “approximation” value obtained by using this truncation technique can be completely unrelated to the actual true utility value.

Example 4.1

The logarithmic utility function $U(X) = \ln(X)$ is one of the most common utility functions. Similarly, the lognormal distribution is one of the most common stochastic models used in economic models. For logarithmic utility and the lognormal distribution pair, we show that the magnitude of the amount thrown away by truncating the Taylor series about the mean after three (or any number of) derivatives bears no relationship to the true expected utility; hence, the error of approximation can be arbitrarily large.[6]

Let X be a lognormally distributed random variable with logarithmic mean and variance $E[\ln(X)] = \eta$ and $E[(\ln(X) - \eta)^2] = \tau^2$, respectively.[7] Then the expected utility is $EU(X) = E[\ln(X)] = \eta$ and the mean (μ), variance (σ^2) and skewness (ρ) measures for

X are

$$\begin{aligned}\mu &= EX = \exp\left(\eta + \frac{\tau^2}{2}\right),\\ \sigma^2 &= E(X-\mu)^2 = \exp(2\eta+\tau^2)[\exp(\tau^2)-1], \text{ and}\\ \rho &= E(X-\mu)^3 = \exp\left(3\eta + \frac{3\tau^2}{2}\right)[\exp(3\tau^2) - 3\exp(\tau^2)+2].\end{aligned}$$

Consequently, the remainder term **R** after truncating the Taylor series expansion for U after three terms and taking expectations is

$$\begin{aligned}\mathbf{R} &= EU(X) - U(\mu) - \frac{U''(\mu)}{2}\sigma - \frac{U'''(\mu)}{6}\rho\\ &= \eta - \ln(\mu) + \frac{\sigma^2}{2\mu^2} - \frac{\rho}{3\mu^3}\\ &= -\frac{\tau^2}{2} + \frac{\exp(\tau^2)-1}{2} - \frac{\exp(3\tau^2) - 3\exp(\tau^2)+2}{3}.\end{aligned}$$

Note that the preceding two central moments are solely a function of the scale parameter τ, so that the remainder term is strictly a function of τ. By appropriate choice of τ or $\eta = EU$, the absolute or relative errors of approximation (that is, **R** or **R**/*EU*) can be made as large or as small as desired. Thus, the truncation of the Taylor series may, in fact, yield a very poor approximation of the true expected utility.[8]

5. A reexamination of empirical results regarding risk and return relations

The previous sections concern theoretical relationships between moment preferences and investor choice. The question now becomes an empirical one. Here we show that the current literature that empirically examines this subject is theoretically flawed.

Researchers sometimes refer to empirical studies that claim to have found a positive (or in some cases a negative)[9] relationship between risk and return. Attempts have been made to empirically estimate the relative importance of the various higher-order moments by using a regression of the mean rate of return on investments with the sample estimates of moments of higher order. For example, empirical studies by Arditti [1967], Levy and Sarnat [1972], Kraus and Litzenberger [1976], and others found the coefficient for the second moment to be positive and statistically significant. This finding was interpreted as indicating that a higher return tends to go together with a prospect that has higher variance—that is, a higher variance seems to be an undesirable property that must be compensated for by having a higher mean. In these studies, the regression coefficient for the third moment (skewness) was negative, and this was interpreted this as evidence for investor preference for a positive asymmetry. The coefficient for the fourth moment (kurtosis) was significant only in a few cases, and the coefficients for higher moments were always insignificant.[10] A logical problem with this interpretation, however, can be asserted. Specifically, if the return distributions were normally distributed, then actual *statistical independence* of the sample mean and variance is a *consequence* and, in fact, is a characterizing property of the normal distribution (see Mathai and Pederzoli [1977]). In this normal distribution situation, then positive coefficients can arise only as a sampling artifact. On the other hand, if the distribution were not normally distributed, then the sample mean and variance are necessarily dependent; thus, correlation may be not so much a statement indicative of a relationship between risk and return as much as a statement concerning the lack of normality of the distributions.

Ruefli [1990] has shown that the direct empirical use of the mean and variance of the same variable as data in the fitting of a model leads to a version of the econometric identification problem in which movements *along a fixed risk-return curve* cannot be distinguished from *shifts in the underlying curve* modeling the relationship. It is possible that the risk-return relationship could be negative in each subinterval but positive over the larger subinterval considered. For example, suppose that during period i with observed data $\mathbf{x}_i$ the relationship between risk $R_i(\mathbf{x}_i)$ and return $Re_i(\mathbf{x}_i)$ is modeled as a function f_i (which may even depend on certain (possibly random) parameters and variables θ_i)—that is, $R_i(\mathbf{x}_i) = f_i[Re_i(\mathbf{x}_i), \theta_i]$. (It does not matter how risk and return are quantified here.) Ruefli's work highlights the fact that it is impossible to tell whether the relationship f_i is constant over i and an observed positive (or negative) relationship between risk $R_i(\mathbf{x}_i)$ and return $Re_i(\mathbf{x}_i)$ is indicative of a true positive (or negative) time invariant relationship f. Instead, it may be the case that each individual relationship f_i is exactly the opposite to that which was postulated to be true—namely, negative (respectively positive)—but the relationship f_i itself changes with i in such a manner that the *global empirical impression* is the opposite of the actual relationship for each i. In a CAPM-type setting, for example, we might have a negative relationship such as $Re_i(\mathbf{x}_i) = \alpha_i - \gamma R_i(\mathbf{x}_i) + \varepsilon$ with $\gamma > 0$ over every subinterval. However, if the intercept term α_i increases with i sufficiently rapidly, then to an analyst performing a more simple regression, such as $Re = \alpha + \gamma R + \varepsilon$, of Re on R using data sampled from different intervals would lead to a positive estimate for γ, indicating exactly the opposite of the true direction for the relationship.[11]

As a final argument against the reliance on empirical analysis for determining the relationship between risk, return, and preference, the recent article by Roll and Ross [1994] shows the impossibility of empirically concluding *any* specific equilibrium relationship exists between the risk measure β and the expected return on an asset. Essentially, they show that if the market portfolio is not *exactly* on the efficiency frontier, then the expected relationship between risk and return is fundamentally ambiguous. Indeed, even an ε departure from the efficiency of the market portfolio makes any relationship (positive, negative, or even zero) unpredictable. Since empirical results *always* by necessity use a market proxy of marketable assets (which can never be *absolutely* guaranteed to be *perfectly* efficient), their results undermine all stated empirical tests of the CAPM type positive relationship. Indeed, the negative relationship between risk and return documented by the Bowman paradox from the strategy literature and the positive relationship between risk and return documented by the finance literature (see note 10) could both be empirically correct according to the ambiguity results of Roll and Ross.

6. Conclusions

Although much has been learned in the past few decades concerning the relationship between moments and utility preferences of decision makers, several old misconceptions still persist. This article has summarized and presented new information and examples on the flaw in the logic behind these misconceptions. Succinctly put, moment orderings are not sufficient for utility preference (although it is a necessary condition for stochastic dominance).

We show that attempts to separately analyze the effect of moment changes on preferences are logically flawed, since in this case *ceteris paribus* is actually *omnibus paribus*. Consequently, we are able to show that the exact opposite relationship can exist between risk, return, skewness, and preference from what continues to show up in the literature. We also evaluate the validity of using empirical arguments to support the use of moment approximations to preference relationships.

Acknowledgment

Dedicated to the memory of our mentor and friend, Bob Witt.

Notes

1. This class of utility functions contains the utility functions commonly used in economics and finance (see Brockett and Golden [1987]).
2. The other two routes involve assuming multivariate normality for returns and arguing that a von Neumann-Morgenstern utility maximizing "rational" decision maker will then choose according to means and variances or appealing to empirical results that have found the desired relationship between mean and variance.
3. Examples in this section were constructed using the theory of Tchebychev functions (cf. Karlin and Studden [1966] as used in Brockett and Cox [1985] and Brockett and Kahane [1987]).
4. Actually, although bounded random variables are assumed in Theorem 1, the results generally apply to random variables for which the moment-generating function exists. A similar proof can be adapted to this situation. Of course, if the random variables are not uniquely determined by their moments (as happens with the lognormal distribution), then no results concerning moment preference and expected utility can be expected to hold since we can construct two variables with all *identical* moments but with different expected utilities.
5. It can be proven that this commonly used pair also exhibits an interesting "nondiversification" property for portfolio selection, since expected return is the only characteristic that appears in the expected utility.
6. This same pair of random choice distribution (lognormal) and decision-maker preference function (logarithmic utility) was also discussed by Feldstein [1969] in a different context. He showed that for this choice pair, the mean-standard deviation indifference curves for a risk averter need not be convex downward although upwardly sloping.
7. The parameters η and τ^2 represent the location and scale parameters for the associated normal distribution for X.
8. This is actually true no matter how many terms are taken in the Taylor expansion. To see this, write $X = \exp(\tau Z + \eta)$ where Z is standard normal. The nth term in the Taylor series involves $E[\exp(\tau Z + \eta) - \exp(\tau + \eta^2/2)]^n/\mu^n = E[\exp(\tau Z) - \exp(\eta^2/2)]^n/\exp(n\eta^2/2)$, which is solely a function of τ and is unrelated to the true expected utility η. For further discussion of this point, see Loistl [1976]. Loistl not only provides a number of examples of expected utility approximations obtained by truncating a Taylor expansion but also shows that the approximation actually worsens as the number of terms in the Taylor series increases.
9. The strategic management literature contains many studies that assert a negative empirical relationship exists between risk and return (that is, the so-called Bowman paradox). See Bowman [1980] and Ruefli [1990] for examples and discussion.

10. As Sections 3 and 4 make apparent, it is unclear how to interpret these empirical results since the logic of regressing the sample higher-order moments on the sample mean is fraught with theoretical difficulties and internal contradictions. Moreover, from Section 3, it is clear that *all* moments are interdependent (since one moment cannot be adjusted without simultaneously changing the other moments in one direction or another), while truncating the Taylor series for use in a regression equation can lead to the difficulties pointed out in Section 4.
11. An analogous situation in statistics that uses probabilities instead of rates or expectations is known as the Simpson's paradox. This paradox exhibits that it is possible for all the conditional probabilities to satisfy $P[A \mid B_i] > P[C \mid B_i]$ for all i, and yet the global (unconditional) relationship may be exactly the reverse: $P[A] < P[C]$ (see Freedman, Pisani, and Purves [1978]).

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