The Demand for Insurance

James R. Garven*

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ABSTRACT: This note provides a simple model of the demand for insurance. We derive the insurance demand equation for a risk-averse individual with logarithmic utility. Comparative statics are derived, and simple proofs of the Bernoulli principle and the Arrow theorem are shown.

INTRODUCTION

This note provides a simple, "single risk" model of the demand for insurance.[1] We derive the insurance demand equation for a risk-averse individual with logarithmic utility. Comparative statics are obtained, and simple proofs of the Bernoulli principle[2] and the Arrow theorem[3] are shown.

The note is organized as follows. In the next section, we outline the insurance decision for an arbitrarily risk-averse individual and prove the Bernoulli principle. In the third section, we derive an insurance demand equation based upon the logarithmic utility function. In the fourth section of the note, comparative statics are derived based upon this utility function. The note concludes with a simple proof of the Arrow (1963) theorem based upon Gerber and Pafumi's (1998) proof of the optimality of stop-loss reinsurance contracts.

AN EXPECTED UTILITY MODEL OF THE DEMAND FOR INSURANCE

An individual has wealth W_0 and will suffer a loss L with probability π. Thus she owns the lottery $(\langle W_0 - L, W_0\rangle, \langle \pi, 1-\pi\rangle)$. She can take out insurance, in which case she must

* James R. Garven is the Frank S. Groner Memorial Chair in Finance and Professor of Finance and Insurance at Baylor University's Hankamer School of Business, Department of Finance, Insurance and Real Estate, Baylor University, One Bear Place #98004, Waco, TX 76798-8004. Phone: (254) 710-6207; e-mail: James_Garven@baylor.edu; web: http://garven.com.

[1]By "single risk", we mean that the insurance decision can be made without considering the possibility of other exogenous or endogenous "background" risks. Exogenous background risks include risks to wealth that are outside the individual's control; e.g., the effect of globalization on financial market returns. Endogenous background risks include risks due to informational asymmetries; e.g., moral hazard and adverse selection. Schlesinger (2000) provides an excellent survey of the multiple- as well as single-risk literature on the demand for insurance.

[2]The Bernoulli principle states that a risk-averse individual will fully insure when the premium is actuarially fair; viz., when it has no loading and is therefore equal to the expected value of loss.

[3]The Arrow Theorem is one of the best-known results in insurance economics. It implies that the optimal insurance contract "…will take the form of 100 per cent coverage above a deductible minimum" (see Arrow (1963, p. 969)).

pay a premium $P = pC$, where p is the premium rate and C is the level of coverage. Thus this individual may exchange the lottery she owns for the lottery

$$(\langle W_0 - pC - L + C,\ W_0 - pC\rangle, \langle \pi, 1\text{-}\pi\rangle). \tag{1}$$

Consider a special case of this lottery, where she full insures risk; i.e., where $C = L$. With full insurance, state contingent wealth is $W_0 - pC$ regardless of whether a state contingent loss occurs; thus she exchanges an uncertain loss (L) for a certain loss (pC).

We assume that this individual has a von Neumann-Morgenstern utility function $U(W)$. Thus $U(W)$ is continuous and twice differentiable; i.e., marginal utility is positive and decreasing in wealth. Given these assumptions, insurance will be purchased if and only if a C exists such that the expected utility of being insured is higher than the expected utility of being uninsured; i.e.,

$$\pi U(W_0 - pC - L + C) + (1 - \pi)U(W_0 - pC) > \pi U(W_0 - L) + (1-\pi)U(W_0). \tag{2}$$

Next, we analyze the optimal insurance decision by solving the problem

$$\max_C E(U(W)) = \pi U(W_0 - pC - L + C) \ + \ (1-\pi)U(W_0 - pC). \tag{3}$$

In order to maximize expected utility, we must solve the first order condition:[4]

$$\pi(1-p)U'(W_0 - pC - L + C) = p(1-\pi)U'(W_0 - pC). \tag{4}$$

Next, we are ready to prove the Bernoulli principle, which states that a risk-averse individual will fully insure if the insurance premium is actuarially fair.

Bernoulli Principle. Suppose insurance is actuarially fair; i.e., $p = \pi$. Substituting π in place of p in equation (4) and simplifying yields equation (5):

$$U'(W_0 - pC - L + C) = U'(W_0 - pC) \Rightarrow W_0 - pC - L + C = W_0 - pC \tag{5}$$

In order for equation (5) to obtain, it must be the case that $C = L$. Therefore, if the insurance premium is actuarially fair, then full coverage is optimal.

INSURANCE DEMAND EQUATION

Next, we derive the insurance demand equation for the case of logarithmic utility.[5] The first order condition given in equation (4) implies that

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[4] Note that the second order condition for a maximum obtains: $d^2E(U(W))/dC^2 = \pi(1-p)^2U''(W_0 - pC - L + C)$ $+(1-\pi)p^2U''(W_0 - pC) < 0$.

[5] The logarithmic utility function was selected because of its analytic tractability. The reader should also note that the

$$\frac{\pi(1-p)}{W_0 - L + (1-p)C} = \frac{p(1-\pi)}{W_0 - pC}. \tag{6}$$

Solving equation (6) for C,[6] we find that

$$C = \frac{(\pi - 1)pL + (p - \pi)W_0}{p(p-1)}. \tag{7}$$

COMPARATIVE STATICS

Effect of Changes in Initial Wealth

An interesting question relates to the effect of changes in initial wealth on the demand for insurance. This relationship can be analyzed by differentiating the optimal value for C given by equation (7) with respect to W_0, resulting in the following equation:

$$\frac{\partial C}{\partial W_0} = \frac{p - \pi}{p(p-1)}. \tag{8}$$

If insurance is actuarially fair; i.e., $p=\pi$, then changes in initial wealth do not affect insurance demand, since full coverage ($C=L$) is optimal, irrespective of the value for W_0! If insurance is unfair; i.e., $p>\pi$, then the demand for insurance is inversely related to the level of initial wealth.

Effect of Changes in the Probability of Loss

Next, we test the relationship between the optimal level of insurance coverage and the probability of loss by differentiating C with respect to π:

$$\frac{\partial C}{\partial \pi} = \frac{Lp - W_0}{p(p-1)}. \tag{9}$$

Since one cannot spend more than initial wealth on insurance, the numerator of this ratio must be negative. We have already determined that the denominator is negative. This implies that the demand for insurance is higher, the higher the probability of loss.

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logarithmic utility function implies decreasing absolute risk aversion and constant relative risk aversion; these behavioral assumptions are valid in terms of both intuition and empirical evidence.

[6] Note that if $p = \pi$, then equation (8) simplifies to $C = L$; i.e., a person with logarithmic utility will fully insure if the insurance premium is actuarially fair.

Consequently, there is a positive relationship between the optimal level of insurance coverage and the probability of loss.

Effect of Changes in Loss Severity

What happens to the optimal level of insurance coverage if the severity of a loss changes? We find the answer to this question by differentiating C with respect to L, resulting in the following equation:

$$\frac{\partial C}{\partial L} = \frac{\pi - 1}{p - 1}. \tag{10}$$

We have previously shown that both the numerator and denominator of this ratio are negative. Consequently, the demand for insurance is positively related to loss severity.

Effect of Changes in the Insurance Premium

Finally, we test the relationship between the optimal level of insurance coverage and the insurance premium by differentiating C with respect to p:

$$\frac{\partial C}{\partial p} = -\frac{Lp^2(\pi - 1) + (p^2 + \pi - 2p\pi)W_0}{(p-1)^2 p^2} = -\frac{\pi(Lp^2 + W_0) + p^2(W_0 - L) - 2p\pi W_0}{(p-1)^2 p^2}. \tag{11}$$

A priori, we expect that the demand for insurance is inversely related to the insurance premium. In equation (11), the denominator is unambiguously positive. In the numerator, the first term ($\pi(Lp^2 + W_0)$) is unambiguously positive, the second term $p^2(W_0 - L) \geq 0$ (with the equality holding only when the entire initial wealth W_0 is at risk), and the third term ($-2p\pi W_0$) is unambiguously negative. Therefore, in order for the demand for insurance to be positively related to the premium, this may only occur when $2p\pi W_0 > \pi(Lp^2 + W_0) + p^2(W_0 - L)$. A positive relationship between C and p would imply that insurance is a Giffen good.[7] However, Hoy and Robson (1981) have shown that insurance cannot be a Giffen good if the coefficient of relative risk aversion is less than or equal to one. Since the coefficient of relative risk aversion for the logarithmic utility function is equal to one,[8] this implies that the sign of equation (11) must be negative; i.e., the demand for

[7] A Giffen good is an "inferior" good with an income effect larger than the substitution effect, so that when the premium decreases, quantity demanded also decreases. In equation (12), one can interpret $2p\pi W_0$ as the income effect, and $\pi(Lp^2 + W_0) + p^2(W_0 - L)$ as the substitution effect.

[8] Since $U(W) = \ln W$, this implies that the coefficient of absolute risk aversion $R_A(W) = -U_{WW}/U_W = 1/W$, and the coefficient of relative risk aversion $R_R(W) = -WU_{WW}/U_W = 1$.

insurance is inversely related to the insurance premium.

OPTIMAL AMOUNT OF DEDUCTIBLE

In the appendix to his famous paper entitled "Uncertainty and the Welfare Economics of Medical Care", Arrow (1963) proves the following proposition, commonly referred to as the Arrow Theorem:

> "Proposition 1: If an insurance company is willing to offer any insurance policy against loss desired by the buyer at a premium which depends only on the policy's actuarial value, then the policy chosen by a risk-averting buyer will take the form of 100 per cent coverage above a deductible minimum. Note: The premium will, in general, exceed the actuarial value; it is only required that two policies with the same actuarial value will be offered by the company for the same premium."

Our alternative proof of the Arrow theorem is based upon Gerber and Pafumi's (1998) proof of the optimality of stop-loss reinsurance contracts.[9] Consider the following two insurance policies:

1. A policy with deductible *d*. In the event of a claim, the indemnity *I*(*L,d*) is of the form $I(L,d) = Max[L-d,0]$.

2. A "general" policy which pays indemnity *I*(*L*), where $0 \le I(L) \le L$. In other words, this policy subsumes the deductible policy, but can also accommodate other types of contract features such as coinsurance and upper limits in addition to or instead of the deductible.

As noted in Arrow's Proposition 1, these policies are assumed to have the same actuarial value and to sell for the same premium. The deductible policy will be preferred to the general policy if the following condition holds:[10]

$$E\big(U(W_0 - L + I(L))\big) \le E\big(U(W_0 - L + Max(L-d,0))\big). \tag{12}$$

Following Gerber and Pafumi, the proof begins by making the simple observation that a concave curve is located below its tangents; i.e.,

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[9]Various papers have provided alternative proofs and extensions of the Arrow theorem. Gollier (1992) reviews much of this literature, and more recently, Gollier and Schlesinger (1996) showed how to derive the Arrow theorem by applying arguments based upon second order stochastic dominance.

[10] Since these policies are assumed to sell for the same premium, we omit the premium from both sides of the inequality given by (12).

$$U(y) \le U(x) + U'(x)(y - x). \tag{13}$$

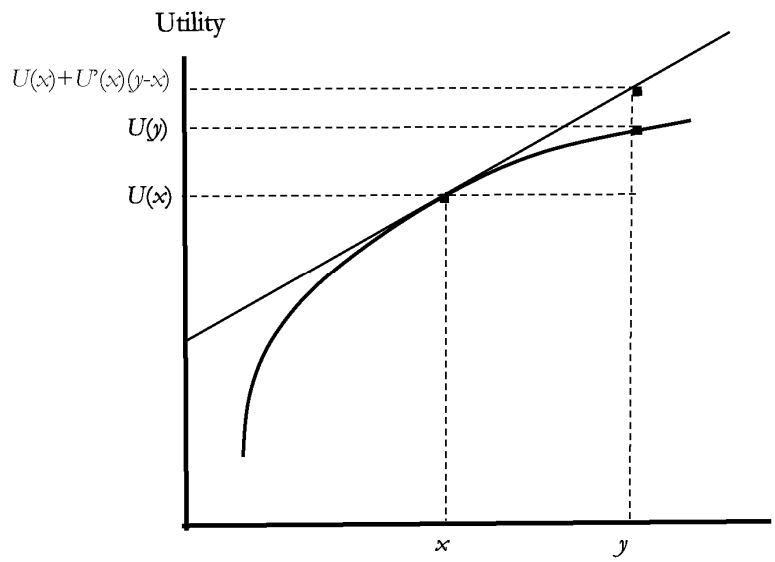

Figure 1. Graphical Illustration of Inequality (13).

The relationship between $U(y)$ and $U(x)$ shown in inequality shown in (13) is graphically represented in Figure 1. Next, let $y = W_0 - L + I(L)$ and $x = W_0 - L + Max(L - d, 0)$. Substituting these definitions for y and x into (13) yields (14):

$$U(W_0 - L + I(L)) \le U(W_0 - L + Max(L - d, 0)) + U'(x)(I(L) - Max(L - d, 0)). \tag{14}$$

Since both policies have the same actuarial value, this implies $E\big((I(L) - Max(L - d, 0))\big) = 0$. Consequently, when we calculate the expected value of both sides of equation (14), we are left with equation (12). In other words, the Arrow theorem obtains.

Now that we have shown that the optimal insurance contract "…will take the form of 100 per cent coverage above a deductible minimum", we conclude this note by showing that the deductible must be nonzero if the premium includes a nonzero proportional loading factor. We do this by retracing our previous analysis; i.e., we assume that an individual with wealth W_0 has the probability π of suffering a loss L. She can take out a deductible insurance policy, in which case she must pay a premium $P = (1 + \lambda)\pi(L - d)$, where d is the deductible and λ is the (non-negative) proportional premium loading. Thus she may exchange her uninsured lottery given by $(\langle W_0 - L, W_0\rangle, \langle \pi, 1-\pi\rangle)$ with an insurance lottery given by $(\langle W_0 - P - d, W_0 - P\rangle, \langle \pi, 1-\pi\rangle)$. Her insurance decision therefore requires solving equation (15):

$$\max_d E(U(W)) = \pi U(W_0 - (1 + \lambda)\pi(L - d) - d) \ + \ (1 - \pi)U(W_0 - (1 + \lambda)\pi(L - d)). \tag{15}$$

The first order condition is given by (16):

$$(1-\pi)\pi(1+\lambda)U'\big(W_0-(1+\lambda)\pi(L-d)\big)=(1-\pi(1+\lambda))\pi U'\big(W_0-(1+\lambda)\pi(L-d)-d\big). \quad (16)$$

Suppose $\lambda = 0$. Then (16) simplifies into the following expression:

$$U'\big(W_0-\pi(L-d)\big)=U'\big(W_0-\pi(L-d)-d\big). \quad (17)$$

By inspection, d must be 0 if $\lambda = 0$. However, if $\lambda > 0$, then d must be greater than 0; otherwise, the marginal utilities of state-contingent wealth in the loss and no-loss states cannot be equal. This latter point becomes more apparent with a specific parameterization for the utility function. For example, suppose that utility is logarithmic. Then the first order condition is given by (18):

$$\frac{\big(1-\pi\big)\pi(1+\lambda)}{W_0-\pi(1+\lambda)(L-d)}=\frac{\big(1-\pi(1+\lambda)\big)\pi}{W_0-\pi(1+\lambda)(L-d)-d} \quad (18)$$

Solving (18) for d, we obtain equation (19):

$$d=\frac{\lambda\big(\pi(1+\lambda)L-W_0\big)}{(1+\lambda)\big(\pi(1+\lambda)-1\big)}. \quad (19)$$

The optimal deductible therefore depends upon the premium loading λ, the probability of loss π, the level of initial wealth W_0, and the loss severity L. The comparative statics for the optimal deductible are as follows:

- $\dfrac{\partial d}{\partial \lambda}=\dfrac{W_0}{(1+\lambda)^2}+\dfrac{(\pi-1)\pi(L-W_0)}{(\pi(1+\lambda)-1)^2}>0$; i.e., the optimal deductible is positively related to the premium loading,
- $\dfrac{\partial d}{\partial \pi}=\dfrac{\lambda\big(W_0-L\big)}{(\pi(1+\lambda)-1)^2}>0$; i.e., the optimal deductible is positively related to the probability of loss,
- $\dfrac{\partial d}{\partial W_0}=\dfrac{\lambda}{\big(1-\pi(1+\lambda)\big)\big(1+\lambda\big)}>0$; i.e., the optimal deductible is positively related to the level of initial wealth, and
- $\dfrac{\partial d}{\partial L}=\dfrac{\pi\lambda}{\big(\pi(1+\lambda)-1\big)}<0$; i.e., the optimal deductible is inversely related to loss severity.

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APPENDIX A

Numerical Comparative Static Analysis of the Demand for Insurance

This appendix provides a simple numerical example of the comparative statics of the demand for insurance, based upon equation (7) in the paper. Suppose W_0 = \$100, L = \$50, π = .50, and p = .50. Inputting these parameters into equation (7) yields

$$C = \frac{(.5-1).5(50)}{.5(.5-1)} = \$50. \tag{A1}$$

Of course, this result (full coverage) is to be expected, since the Bernoulli principle implies that whenever $p = \pi$, then $C = L$.

Suppose W_0, L and π do not change, but the insurer decides to charge a 20% premium loading; i.e., now, p = .60. Then

$$C = \frac{(.5-1).6(50)+.1(100)}{.6(.6-1)} = \$20.83. \tag{A2}$$

Here, the level of coverage falls because insurance has become more expensive. The individual is only willing to purchase partial coverage because the increase in premium limits the utility gain obtained from the purchase of an insurance policy.

Suppose L and π do not change, but W_0 increases from \$100 to \$120 and the premium rate is maintained at p = .60. Then

$$C = \frac{(.5-1).6(50)+.1(120)}{.6(.6-1)} = \$12.50. \tag{A3}$$

The level of coverage falls with an increase in initial wealth because of diminishing marginal utility. Basically, the utility loss from limiting insurance coverage is less severe for a "wealthy" person than it is for an otherwise identical "poor" person.

Suppose W_0 = \$120, π = .50, p = .60, and L increases from \$50 to \$60. Then

$$C = \frac{(.5-1).6(60)+.1(120)}{.6(.6-1)} = \$25. \tag{A4}$$

The level of coverage increases because severity has increased; the larger loss implies that the individual gains more utility from transferring risk to an insurer, even though the premium is not actuarially fair.

Finally, suppose W_0 = \$120, L = \$60, p = .60, and π increases from .50 to .60. Then

$$C = \frac{(.6-1).6(60)+.1(120)}{.6(.6-1)} = \$60. \tag{A5}$$

The level of coverage increases because there is a higher risk of loss. Also, we know from the Bernoulli principle since $p = \pi$, full coverage ($C = L$) is optimal.

APPENDIX B

Numerical Comparative Static Analysis of the Optimal Deductible

This appendix provides a simple numerical example of the comparative statics of the optimal insurance deductible, based upon equation (19) in the paper. Suppose $W_0 = \$100$, $L = \$50$, $\pi = .50$, and $\lambda = .20$. Inputting these parameters into equation (19) yields:

$$d = \frac{.2\left(.5(1.2)50-100\right)}{(1.2)\left(.5(1.2)-1\right)} = \$29.17. \tag{B1}$$

Our comparative static analysis indicated that the optimal deductible is positively related to the premium loading. Suppose $\lambda = .40$ rather than .2. Then

$$d = \frac{.4\left(.5(1.4)50-100\right)}{(1.4)\left(.5(1.4)-1\right)} = \$61.90. \tag{B2}$$

The optimal deductible is also positively related to the probability of loss and the level of initial wealth. In (B3) and (B4), we recalibrate the numerical example back to the "base case" shown in (B1), and then increase π and L to .6 and \$150 respectively:

$$d = \frac{.2\left(.6(1.2)50-100\right)}{(1.2)\left(.6(1.2)-1\right)} = \$38.10. \tag{B3}$$

$$d = \frac{.2\left(.5(1.2)50-150\right)}{(1.2)\left(.5(1.2)-1\right)} = \$50. \tag{B4}$$

Finally, our comparative static analysis indicated that the optimal deductible is inversely related to the loss severity. In (B5), we recalibrate the numerical example back to the "base case" shown in (B1), and then increase L from \$50 to \$75:

$$d = \frac{.2\left(.5(1.2)75-100\right)}{(1.2)\left(.5(1.2)-1\right)} = \$22.92. \tag{B5}$$