

Derivation and Comparative Statics of the Black-Scholes Call and Put Option Pricing Formulas

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Abstract

This paper provides an alternative derivation of the Black-Scholes call and put option pricing formulas using an integration rather than differential equations approach. The economic and mathematical structure of these formulas is discussed, and comparative statics are derived.

1 Introduction

The purposes of this paper are: (1) to provide a concise overview of the economic and mathematical assumptions needed to derive the [Black and Scholes \(1973\)](#) call and put option pricing formulas; (2) to provide an alternative derivation and comparative statics analysis of these formulas.

This paper is organized as follows. In the next section, we discuss the economic and mathematical structure of the Black-Scholes model. In the third section, we provide an alternative derivation of the Black-Scholes option pricing formulas based upon an integration rather than differential equations approach. The advantage of the integration approach lies in its simplicity, as only basic integral calculus is required. The integration approach also makes the link between the economics and mathematics of the call and put option formulas

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more transparent for most readers. This paper concludes with comparative statics analyses of these formulas.

2 Economic and Mathematical Structure of the Black-Scholes Model

A particularly important economic insight contained in Black and Scholes' seminal paper ([Black and Scholes \(1973\)](#)) is the notion that an investor can create a riskless portfolio by dynamically hedging a long position in a stock with a short position in a European call option written on that stock.¹ In order to prevent arbitrage, the expected return on such a portfolio must be the riskless rate of interest. [Cox and Ross \(1976\)](#) note that since the creation of such a portfolio places no restrictions on investor preferences beyond nonsatiation, the valuation relationship between an option and its underlying asset is "risk neutral" in the sense that it does not depend upon investor risk preferences. Therefore, for a given price of the underlying asset, a call option written against that asset will trade for the same price in a risk neutral economy as it would in a risk averse or risk loving economy. Consequently, options may be priced *as if* they are traded in risk neutral economies.

Black and Scholes assume that stock prices change continuously according to the Geometric Brownian Motion equation; i.e.,

$$dS = \mu S dt + \sigma S dz. \tag{1}$$

Equation (1) is a stochastic differential equation because it contains the Wiener process $dz = \epsilon\sqrt{dt}$, where ϵ is a standard normal random variable with mean $E(\epsilon) = 0$ and variance $Var(\epsilon) = 1$. The differential dS corresponds to the stock price change per dt unit of time, S

¹While Black and Scholes consider the case of a long stock/short option trading strategy in their paper, a riskless portfolio can also obviously be created by dynamically hedging a short position in the underlying asset with a long position in a European call option. The Black-Scholes trading strategy synthetically replicates a *long* riskless bond position, whereas a short stock/long option trading strategy synthetically replicates a *short* riskless bond position.

is the current (date t) stock price,² μ is the (annualized) expected return, and σ represents (annualized) volatility. The Geometric Brownian Motion equation is often referred to as an *exponential* stochastic differential equation because its solution is an exponential function; specifically, $S_T = Se^x$, where S_T represents the stock price $\tau = T - t$ periods from now, $\tau \geq 0$, and $x = (\mu - \sigma^2/2)\tau + \epsilon\sigma\sqrt{\tau}$. Since ϵ is normally distributed, so is x ; thus, S_T is lognormally distributed.³ At time T , the mean of S_T is $E(S_T) = Se^{\mu\tau}$, and its variance is $\sigma_{S_T}^2 = S^2e^{2\mu\tau}(e^{\sigma^2\tau} - 1) = (E(S_T))^2(e^{\sigma^2\tau} - 1)$. Since S_T is lognormally distributed, S_T/S is also lognormally distributed; therefore, $\ln S_T/S$ is normally distributed. Furthermore, $\ln S_T/S$ corresponds to the τ period *continuously compounded* rate of return on the stock.⁴

The next step involves determining the stochastic process which governs the price dynamics for the call option. Since we have a model of the price dynamics for the underlying stock given by equation (1), we can apply Itô's Lemma in order to obtain such an equation for the call option.⁵ Thus,

$$dC = \frac{\partial C}{\partial t}dt + \frac{\partial C}{\partial S}dS + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}dS^2. \quad (2)$$

Here, dC corresponds to the call option price change per dt unit of time.

Next, we simplify the right hand side of equation (2). Since the third term in that equation is a function of dS^2 , we square the right hand side of equation (1) and obtain $dS^2 = (\mu Sdt + \sigma Sdz)^2 = \mu^2 S^2 dt^2 + \sigma^2 S^2 dt + 2\mu\sigma Sdt^{3/2}$. However, since the first and third terms of this equation involve dt raised to powers greater than 1, this implies that $dS^2 =$

²Throughout this paper, whenever a "current" market value is referenced, this is a reference to market value at date t .

³The lognormal distribution is a particularly suitable candidate for modeling stock prices. Besides being mathematically tractable, the lognormal distribution generates price patterns that resemble real world stock price patterns. Furthermore, under the lognormal distribution, stock prices are non-negative, being bounded from below at zero and unbounded from above.

⁴This highlights yet another advantage of assuming that stock prices are lognormally distributed. Lognormally distributed prices imply that continuously compounded stock returns ($x = \ln(S_T/S)$) are normally distributed, thus resembling real world returns (in the sense that realized returns can be negative, zero, or positive).

⁵Essentially, Itô's Lemma is based upon a second order Taylor series expansion of $f(S, t)$, where f corresponds to the price of a derivative security based upon S . See Pennacchi (2008), pp. 238-240 for a heuristic derivation of Itô's Lemma.

$\sigma^2 S^2 dt$.⁶ Substituting this expression for dS^2 into equation (2) yields equation (3):

$$dC = \frac{\partial C}{\partial t} dt + \frac{\partial C}{\partial S} dS + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} dt. \quad (3)$$

Suppose that at time t , we construct a hedge portfolio consisting of one long call option position worth $C(S, t)$ and a short position in some quantity Δ_t of the underlying asset worth S_T per share. We express the hedge ratio Δ_t as a function of t because the portfolio will be *dynamically hedged*; i.e., as the price of the underlying asset changes through time, so will Δ_t .⁷ Then the value of this hedge portfolio is $V = C(S, t) - \Delta_t S$, which implies

$$dV = dC - \Delta_t dS = \underbrace{\left(\frac{\partial C}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} \right) dt}_{\text{deterministic}} + \underbrace{\left(\frac{\partial C}{\partial S} - \Delta_t \right) dS}_{\text{stochastic}}. \quad (4)$$

Note that there are stochastic as well as deterministic components on the right-hand side of equation (4). The deterministic component is represented by the first product involving dt , whereas the stochastic component is represented by the second product involving dS . However, by setting Δ_t equal to $\frac{\partial C}{\partial S}$, the stochastic component disappears since $\frac{\partial C}{\partial S} - \Delta_t = 0$, leaving:

$$dV = dC - \Delta_t dS = \left(\frac{\partial C}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} \right) dt. \quad (5)$$

Since this is a perfectly hedged portfolio, it has no risk. In order to prevent arbitrage, the hedge portfolio must earn the riskless rate of interest r ; i.e.,

$$dV = rV dt. \quad (6)$$

We will assume that $\Delta_t = \frac{\partial C}{\partial S}$, so $V = C(S, t) - \frac{\partial C}{\partial S} S$. Substituting this into the right-hand

⁶Pennacchi (2008) makes the following very important observation about Wiener processes (cf. footnote 12 on page 240): “. . . it may be helpful to remember that in the continuous-time limit $dz^2 = dt$, but $dzdt = 0$, and $dt^n = 0$ for $n > 1$.”

⁷Intuitively, as the share price increases (decreases), then Δ_t must also increase (decrease) in order to form a perfect hedge, since changes in the price of the call option will more (less) closely mimic changes in the share price as the share price increases (decreases).

side of equation (6) and equating the result with the right-hand side of equation (5), we obtain:

$$r \left(C - S \frac{\partial C}{\partial S} \right) dt = \left(\frac{\partial C}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} \right) dt. \quad (7)$$

Dividing both sides of equation (7) by dt and rearranging results in the Black-Scholes (non-stochastic) partial differential equation:⁸

$$rC = \frac{\partial C}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + rS \frac{\partial C}{\partial S}. \quad (8)$$

Equation (8) shows that the valuation relationship between a call option and its underlying asset is deterministic because dynamic hedging enables the investor to be perfectly hedged over infinitesimally small units of time. Since risk preferences play no role in this equation, this implies that the price of a call option may be calculated *as if* investors are risk neutral.

Assuming there are τ periods until option expiration occurs, the value of a call option at date t (C) must satisfy equation (8), subject to the boundary condition on the option's payoff at expiration; i.e., $C_T = \text{Max}[S_T - K, 0]$, where K represents the option's exercise price and S_T represents the stock price at expiration. Black and Scholes transform their version of equation (8) into the heat transfer equation of physics,⁹ which allows them to (quite literally) employ a "textbook" solution procedure taken from Churchill's (1963) classic introduction to Fourier series and their applications to boundary value problems in partial differential equations of engineering and physics. This solution procedure results in the following equation for the value of a European call option:

⁸Note that equation (8) in this paper and equation (7) in the original Black-Scholes paper (cf. Black and Scholes (1973), available on the web at <http://www.jstor.org/stable/1831029>) are the same equation (although the notation is marginally different).

⁹Wilmott (2001, p. 156) notes that heat transfer equations date back to the beginning of the 19th century, and have been used to model a diverse set of physical and social phenomena, such as the flow of heat from one part of an object to another, electrical activity in the membranes of living organisms, the dispersion of populations, the formation of zebra stripes, and the dispersion of air and water pollution, among other things.

$$C = SN(d_1) - Ke^{-r\tau}N(d_2), \tag{9}$$

where

$$d_1 = \frac{\ln(S/K) + (r + .5\sigma^2)\tau}{\sigma\sqrt{\tau}};$$

$$d_2 = d_1 - \sigma\sqrt{\tau};$$

σ^2 = variance of underlying asset's rate of return; and

$N(z)$ = standard normal distribution function evaluated at z .

Equation (9) implies that the price of the call option depends upon five parameters; specifically, the current (i.e., date t) stock price S , the exercise price K , the riskless interest rate r , the time to expiration τ , and the volatility of the underlying asset σ .

If it is not possible to form a riskless hedge portfolio, then other more restrictive assumptions are needed in order to obtain a risk neutral valuation relationship between an option and its underlying asset. Rubinstein (1976) has shown that the Black-Scholes option pricing formulas also obtain under the following set of assumptions:

1. The conditions for aggregation are met so that securities are priced *as if* all investors have the same characteristics as a representative investor;
2. The utility function of the representative investor exhibits constant relative risk aversion (CRRA);
3. The return on the underlying asset and the return on aggregate wealth are bivariate lognormally distributed;
4. The return on the underlying asset follows a stationary random walk through time; and
5. The riskless rate of interest is constant through time.

Collectively, assumptions 1, 2, and 3 constitute sufficient conditions for valuing the option *as if* both it and the underlying asset are traded in a risk neutral economy. Assumptions 4 and 5 constitute necessary conditions for the Black-Scholes option pricing formulas, since they guarantee that the riskless rate of interest, the rate of return on the underlying asset, and the variance of the underlying asset will be proportional to time.¹⁰

¹⁰Brennan (1979) shows that if the return on the underlying asset and the return on aggregate wealth are

3 Black-Scholes Option Pricing Formula Derivations

Rubinstein (1987) notes that the Black-Scholes call option pricing formula may be derived by integration if one assumes that the probability distribution for the underlying asset is lognormal. Fortunately, as noted earlier, this assumption is implied by the Geometric Brownian Motion equation given by equation (1). Therefore, our next task is to derive the Black-Scholes call option pricing formula by integration. Once we obtain the Black-Scholes call option pricing formula, the put option pricing formula follows directly from the put-call parity theorem.

The current (i.e., date t) value (C) of a European call option that pays $C_T = \text{Max}[S_T - K, 0]$ τ periods from now is given by the following equation:

$$C = V(C_T) = V(\text{Max}[S_T - K, 0]), \quad (10)$$

where $V(\cdot)$ represents the valuation operator. Having established in equation (8) that a risk neutral valuation relationship exists between the call option and its underlying asset, it follows that we can price the option *as if* investors are risk neutral. Therefore, the valuation operator $V(\cdot)$ determines the current price of a call option by discounting the risk neutral expected value of the option's payoff at expiration ($\hat{E}(C_T)$) at the riskless rate of interest, as indicated in equation (11):

$$C = e^{-r\tau} \hat{E}(C_T) = e^{-r\tau} \int_K^{\infty} (S_T - K) \hat{h}(S_T) dS_T, \quad (11)$$

where $\hat{h}(S_T)$ represents the risk neutral lognormal density function of S_T .

bivariate lognormally (normally) distributed, then the representative investor must have CRRA (CARA) preferences. Stapleton and Subrahmanyam (1984) extend Rubinstein's and Brennan's results to a world where payoffs on options depend on the outcomes of two or more stochastic variables.

We begin our analysis by computing the expected value of C_T :

$$E(C_T) = E[\text{Max}(S_T - K, 0)] = \int_K^\infty (S_T - K)h(S_T)dS_T, \quad (12)$$

where $h(S_T)$ represents S_T 's “true” lognormal density function.¹¹ To evaluate this integral, we rewrite it as the difference between two integrals:

$$E(C_T) = \int_K^\infty S_T h(S_T) dS_T - K \int_K^\infty h(S_T) dS_T = E_K(S_T) - K [1 - H(K)], \quad (13)$$

where

$E_K(S_T)$ = the partial expected value of S_T , truncated from below at K ;¹²
 $H(K)$ = the probability that $S_T \leq X$.

Next, we define the τ -period lognormally distributed price ratio as $R_T = S_T/S$. Thus, $S_T = S(R_T)$, and we rewrite equation (13) as

$$\begin{aligned} E(C_T) &= S \int_{K/S}^\infty R_T g(R_T) dR_T - K \int_{K/S}^\infty g(R_T) dR_T \\ &= S E_{K/S}(R_T) - K [1 - G(K/S)], \end{aligned} \quad (14)$$

where

$g(R_T)$ = lognormal density function of R_T ;¹³
 $E_{K/S}(R_T)$ = the partial expected value of R_T , truncated from below at K/S ; and
 $G(K/S)$ = the probability that $R_T \leq K/S$.

Next, we evaluate the right-hand side of equation (14) by considering its two integrals

¹¹The difference between the “risk neutral” $\hat{h}(S_T)$ and “true” $h(S_T)$ density functions is easily explained. Since S_T is a lognormally distributed random variable, $E(S_T) = \int_0^\infty S_T h(S_T) dS_T = S e^{\mu\tau}$. Given that assets in a risk neutral economy are expected to return the riskless rate of interest, this implies that $\hat{E}(S_T) = \int_0^\infty S_T \hat{h}(S_T) dS_T = S e^{r(T-t)}$, which in turn implies that $\mu = r$.

¹²The partial expected value of X for values of X ranging from y to z is $E_y^z(X) = \int_y^z X f(X) dX$. If we replace the y and z limits of integration with $-\infty$ and ∞ , then we have a “complete” expected value, or first moment. See [Winkler, Roodman, and Britney \(1972\)](#) for more details concerning the computation of partial moments for a wide variety of probability distributions.

¹³Note that $g(R_T) = h(S_T)/S$.

separately. First, consider the integral that corresponds to the partial expected value of the terminal stock price, $SE_{K/S}(R_T)$. Let $R_T = e^{k\tau}$, where k is the annualized rate of return on the underlying asset. Since R_T is lognormally distributed, $\ln(R_T) = k\tau$ is normally distributed with density $f(k\tau)$, mean $\mu_k\tau = (\mu - \frac{1}{2}\sigma^2)\tau$ and variance $\sigma^2\tau$.¹⁴ Furthermore, $g(R_T) = (1/R_T) f(k\tau)$.¹⁵ Also, since $R_T = e^{k\tau}$, it follows that $dR_T/dk = \tau e^{k\tau}$, which implies that $dR_T = e^{k\tau}\tau dk$. Consequently, $R_T g(R_T) dR_T = e^{k\tau} f(k\tau)\tau dk$. Thus,

$$SE_{K/S}(R_T) = S \int_{\ln(K/S)}^{\infty} e^{k\tau} f(k\tau)\tau dk = S \frac{1}{\sqrt{2\pi\sigma^2\tau}} \int_{\ln(K/S)}^{\infty} e^{k\tau} e^{-\{.5[(k\tau - \mu_k\tau)^2/\sigma^2\tau]\}} \tau dk. \quad (15)$$

Next, we simplify equation (15)'s integrand by adding the terms in the two exponents, multiplying and dividing the resulting expression by $e^{-.5\sigma^2\tau}$, and rearranging:

$$\begin{aligned} e^{k\tau} e^{-\{.5[(k\tau - \mu_k\tau)^2/\sigma^2\tau]\}} &= e^{-\{.5\tau[(k^2 - 2\mu_k k + \mu_k^2 - 2\sigma^2 k)/\sigma^2]\}} \\ &= e^{-\{.5\tau[(k^2 - 2\mu_k k + \mu_k^2 - 2\sigma^2 k + \sigma^4 - \sigma^4)/\sigma^2]\}} \\ &= e^{-\{.5\tau[(k - \mu_k - \sigma^2)^2 - \sigma^4 - 2\mu_k\sigma^2]/\sigma^2\}} \\ &= e^{(\mu_k + .5\sigma^2)\tau} e^{-\{.5[(k\tau - (\mu_k + \sigma^2)\tau)^2/\sigma^2\tau]\}}. \end{aligned} \quad (16)$$

In equation (16), the term $e^{(\mu_k + .5\sigma^2)\tau} = e^{(\mu - .5\sigma^2 + .5\sigma^2)\tau} = e^{\mu\tau} = E(R_T)$, the mean of the τ -period lognormally distributed price ratio R_T . Therefore, we can rewrite equation (15) with this term appearing outside the integral:

$$\begin{aligned} SE_{K/S}(R_T) &= SE(R_T) \frac{1}{\sqrt{2\pi\sigma^2\tau}} \int_{\ln(K/S)}^{\infty} e^{-\{.5[(k\tau - (\mu_k + \sigma^2)\tau)^2/\sigma^2\tau]\}} \tau dk \\ &= E(S_T) \frac{1}{\sqrt{2\pi\sigma^2\tau}} \int_{\ln(K/S)}^{\infty} e^{-\{.5[(k\tau - (\mu_k + \sigma^2)\tau)^2/\sigma^2\tau]\}} \tau dk. \end{aligned} \quad (17)$$

Next, we let $y = [k\tau - (\mu_k + \sigma^2)\tau]/\sigma\sqrt{\tau}$, which implies that $k\tau = (\mu_k + \sigma^2)\tau + \sigma\sqrt{\tau}y$ and $\tau dk = \sigma\sqrt{\tau}dy$. With this change in variables, $[\ln(K/S) - (\mu_k + \sigma^2)\tau]/\sigma\sqrt{\tau} = -\delta_1$ becomes

¹⁴See Hull (2012), pp. 292-293 and Garven (2017) for a formal analysis (based upon Itô's Lemma) of the relationship between R_T and $\ln(R_T)$.

¹⁵Since $G(R_T) = F(k\tau)$, $dG(R_T)/dR_T = g(R_T)$ and $dF(k\tau)/d(R_T) = (dF(k\tau)/d(k\tau))(dk\tau/d(R_T)) = f(k\tau)(1/R_T)$; thus, $g(R_T) = (1/R_T)f(k\tau)$.

the lower limit of integration. Thus equation (17) may be rewritten as:

$$\begin{aligned}
SE_{K/S}(R_T) &= E(S_T) \int_{-\delta_1}^{\infty} [e^{-.5y^2}/\sqrt{2\pi}] dy \\
&= E(S_T) \int_{-\infty}^{\delta_1} n(y) dy \\
&= E(S_T)N(\delta_1),
\end{aligned} \tag{18}$$

where $n(y)$ is the standard normal density function evaluated at y and $N(\delta_1)$ is the standard normal distribution function evaluated at $y = \delta_1$.

In order to complete our computation of the expected value of C_T , we must evaluate the integral $K \int_{K/S}^{\infty} g(R_T)dR_T$ which appears in equation (14). As noted earlier, since $R_T = e^{k\tau}$ is lognormally distributed, $\ln(R_T) = k\tau$ is normally distributed with density $f(k\tau)$, mean $\mu_k\tau = (\mu - \frac{1}{2}\sigma^2)\tau$ and variance $\sigma^2\tau$. Furthermore, this change in variable implies that $f(k\tau)\tau dk$ must be substituted in place of $g(R_T)dR_T$. Therefore,

$$\begin{aligned}
K \int_{K/S}^{\infty} g(R_T)dR_T &= K \int_{\ln(K/S)}^{\infty} f(k\tau)\tau dk \\
&= K \frac{1}{\sqrt{2\pi\sigma^2\tau}} \int_{\ln(K/S)}^{\infty} e^{-\{.5[(k\tau - \mu_k\tau)^2/\sigma^2\tau]\}} \tau dk.
\end{aligned} \tag{19}$$

Next, we let $z = [k\tau - \mu_k\tau]/\sigma\sqrt{\tau}$, which implies that $k\tau = \mu_k\tau + \sigma\sqrt{\tau}z$ and $\tau dk = \sigma\sqrt{\tau}dz$. With this change in variables, the lower limit of integration becomes $[\ln(K/S) - \mu_k\tau]/\sigma\sqrt{\tau} = -(\delta_1 - \sigma\sqrt{\tau}) = -\delta_2$. Substituting these results into the right-hand side of equation (19) yields equation (20):

$$\begin{aligned}
K \int_{\ln(K/S)}^{\infty} f(k\tau)\tau dk &= K \int_{-\delta_2}^{\infty} [e^{-.5z^2}/\sqrt{2\pi}] dz \\
&= K \int_{-\infty}^{\delta_2} n(z) dz \\
&= KN(\delta_2),
\end{aligned} \tag{20}$$

where $n(z)$ is the standard normal density function evaluated at z and $N(\delta_2)$ is the standard normal distribution function evaluated at $z = \delta_2$. Substituting the right-hand sides of

equations (18) and (20) into the right-hand side of equation (14), we obtain

$$E(C_T) = E(S_T)N(\delta_1) - KN(\delta_1 - \sigma\sqrt{\tau}). \quad (21)$$

Since equation (11) defines the price of a call option as the discounted, risk neutral expected value of the option's payoff at expiration; i.e., $C = e^{-r\tau}\hat{E}(C_T)$, our next task is to compute the risk neutral expected value of the stock price on the expiration date ($\hat{E}(S_T)$) and the risk neutral value for δ_1 (which we will refer to as d_1). In footnote (11), we noted that $\mu = r$ in a risk neutral economy. Since $E(S_T) = Se^{(\mu_k + .5\sigma^2)\tau} = Se^{\mu\tau}$, this implies that $\hat{E}(S_T) = Se^{r\tau}$. In the expression for δ_1 , the term $(\mu_k + \sigma^2)\tau$ appears. In a risk neutral economy, $(\mu_k + \sigma^2)\tau = (\mu - .5\sigma^2 + \sigma^2)\tau = (r + .5\sigma^2)\tau$; therefore, $d_1 = [\ln(S/K) + (r + .5\sigma^2)t]/\sigma\sqrt{\tau}$. Substituting the right-hand side of equation (21) into the right-hand side of equation (11) and simplifying yields the Black-Scholes call option pricing formula:

$$\begin{aligned} C &= e^{-r\tau}\hat{E}(C_T) = e^{-r\tau} [Se^{r\tau}N(d_1) - KN(d_1 - \sigma\sqrt{\tau})] \\ &= SN(d_1) - Ke^{-r\tau}N(d_2). \end{aligned} \quad (22)$$

Now that we have the Black-Scholes pricing formula for a European call option, the put option pricing formula follows directly from the put-call parity theorem. Suppose two portfolios exist, one consisting of a European call option and a riskless discount bond, and the other consisting of a European put option and a share of stock against which both options are written. The call and put both have exercise price K and τ periods to expiration, and the riskless bond pays off K dollars at date T . Then these portfolios' date T payoffs are identical, since the payoff on the first portfolio is $Max(S_T - K, 0) + K = Max(S_T, K)$ and the payoff on the second portfolio is $Max(K - S_T, 0) + S_T = Max(S_T, K)$. Consequently, the current value of these portfolios must also be the same; otherwise there would be a riskless arbitrage opportunity. Therefore, the price of a European put option, P , may be determined as follows:

$$\begin{aligned}
P &= C + Ke^{-r\tau} - S \\
&= SN(d_1) - Ke^{-r\tau}N(d_2) + Ke^{-r\tau} - S \\
&= Ke^{-r\tau} [1 - N(d_2)] - S [1 - N(d_1)] \\
&= Ke^{-r\tau}N(-d_2) - SN(-d_1).
\end{aligned} \tag{23}$$

4 Comparative Statics

As indicated by equations (22) and (23), the prices of European call and put options depend upon five parameters: S , K , t , r , and σ . Next, we provide comparative statics analysis of call and put option prices.

4.1 Relationship between the call and put option prices and the price of the underlying asset (Delta)

Consider the relationship between the price of the call option and the price of the underlying asset, $\partial C/\partial S$:

$$\begin{aligned}
\partial C/\partial S &= N(d_1) + S(\partial N(d_1)/\partial d_1)(\partial d_1/\partial S) - Ke^{-r\tau}(\partial N(d_2)/\partial d_2)(\partial d_2/\partial S) \\
&= N(d_1) + Sn(d_1)(\partial d_1/\partial S) - Ke^{-r\tau}n(d_2)(\partial d_2/\partial S).
\end{aligned} \tag{24}$$

Substituting $d_2 = d_1 - \sigma\sqrt{\tau}$, $\partial d_2/\partial S = \partial d_1/\partial S$, and $n(d_2) = n(d_1 - \sigma\sqrt{\tau})$ into equation (24), we obtain:

$$\begin{aligned}
\partial C/\partial S &= N(d_1) + (\partial d_1/\partial S)[Sn(d_1) - Ke^{-r\tau}n(d_1 - \sigma\sqrt{\tau})] \\
&= N(d_1) + (\partial d_1/\partial S)\frac{1}{\sqrt{2\pi}}[Se^{-.5d_1^2} - Ke^{-r\tau}e^{-.5(d_1 - \sigma\sqrt{\tau})^2}].
\end{aligned} \tag{25}$$

Recall that $d_1 = [\ln(S/K) + (r + .5\sigma^2)\tau]/\sigma\sqrt{\tau}$. Solving for S , we find that $S = Ke^{d_1\sigma\sqrt{\tau} - (r + .5\sigma^2)\tau}$. Substituting this expression for S into the bracketed term on the right-hand side of equation (25) yields

$$\begin{aligned}
\partial C/\partial S &= N(d_1) + (\partial d_1/\partial S) \frac{1}{\sqrt{2\pi}} [K e^{-.5d_1^2} e^{d_1\sigma\sqrt{\tau} - (r+.5\sigma^2)\tau} - K e^{-r\tau} e^{-.5(d_1 - \sigma\sqrt{\tau})^2}] \\
&= N(d_1) + (\partial d_1/\partial S) \frac{1}{\sqrt{2\pi}} [K (e^{-(r+.5\sigma^2)\tau + d_1\sigma\sqrt{\tau} - .5d_1^2} - e^{-(r+.5\sigma^2)\tau + d_1\sigma\sqrt{\tau} - .5d_1^2})]. \quad (26)
\end{aligned}$$

Since the bracketed term on the right-hand side of equation (26) equals zero, $\partial C/\partial S = N(d_1) > 0$; i.e., the price of a call option is positively related to the price of its underlying asset. Furthermore, from our earlier analysis of the Black-Scholes partial differential equation, we know that a dynamically hedged portfolio consisting of one long call position per $\Delta_t = \partial C/\partial S$ short shares of the underlying asset is riskless (cf. equation (5)). Since $\partial C/\partial S = N(d_1)$, $N(d_1)$ is commonly referred to as the option's *delta*, and the dynamic hedging strategy described here is commonly referred to as *delta hedging*.

Intuitively, since the put option provides its holder with the right to sell rather than buy, one would expect an inverse relationship between the price of a put option and the price of its underlying asset. Next, we analyze this relationship by finding $\partial P/\partial S$:

$$\begin{aligned}
\partial P/\partial S &= -N(-d_1) + K e^{-r\tau} (\partial N(-d_2)/\partial d_2) (\partial d_2/\partial S) - S (\partial N(-d_1)/\partial d_1) (\partial d_1/\partial S) \\
&= -N(-d_1) + K e^{-r\tau} n(-d_2) (\partial d_1/\partial S) - S n(-d_1) (\partial d_1/\partial S) \\
&= -N(-d_1) + (\partial d_1/\partial S) [K e^{-r\tau} n(\sigma\sqrt{\tau} - d_1) - S n(-d_1)] \quad (27)
\end{aligned}$$

Given the symmetry of the standard normal distribution about its mean of zero, $S n(-d_1) = S n(d_1)$ and $K e^{-r\tau} n(-d_2) = K e^{-r\tau} n(d_2)$. Since we know from equations (25) and (26) that $S n(d_1) - K e^{-r\tau} n(d_1 - \sigma\sqrt{\tau}) = 0$, it follows that the bracketed term on the right-hand side of equation (27), $K e^{-r\tau} n(\sigma\sqrt{\tau} - d_1) - S n(-d_1) = 0$. Therefore, $\partial P/\partial S = -N(-d_1) < 0$. Thus, our intuition is confirmed; i.e., the price of a put option is inversely related to the price of its underlying asset.

4.2 Relationship between call and put option prices and the exercise price

Consider next the relationship between the price of the call option and the exercise price, $\partial C/\partial K$:

$$\begin{aligned}\partial C/\partial K &= -e^{-r\tau}N(d_2) + S(\partial N(d_1)/\partial d_1)(\partial d_1/\partial K) - Ke^{-r\tau}(\partial N(d_2)/\partial d_2)(\partial d_2/\partial K) \\ &= -e^{-r\tau}N(d_2) + Sn(d_1)(\partial d_1/\partial K) - Ke^{-r\tau}n(d_2)(\partial d_2/\partial K)\end{aligned}\quad (28)$$

Substituting $d_2 = d_1 - \sigma\sqrt{\tau}$, $\partial d_2/\partial K = \partial d_1/\partial K$ and $n(d_2) = n(d_1 - \sigma\sqrt{\tau})$ into equation (28) gives us

$$\partial C/\partial K = -e^{-r\tau}N(d_2) + (\partial d_1/\partial K)[Sn(d_1) - Ke^{-r\tau}n(d_1 - \sigma\sqrt{\tau})] = -e^{-r\tau}N(d_2) < 0. \quad (29)$$

Intuition suggests that since the price of a call option is inversely related to its exercise price, one would expect to find that a positive relationship exists between the price of a put option and its exercise price, which we confirm in equation (30):

$$\begin{aligned}\partial P/\partial K &= e^{-r\tau}N(-d_2) + Ke^{-r\tau}(\partial N(-d_2)/\partial d_2)(\partial d_2/\partial K) - S(\partial N(-d_1)/\partial d_1)(\partial d_1/\partial K) \\ &= e^{-r\tau}N(-d_2) + Ke^{-r\tau}n(-d_2)(\partial d_1/\partial K) - Sn(-d_1)(\partial d_1/\partial K) \\ &= e^{-r\tau}N(-d_2) + (\partial d_1/\partial K)[Ke^{-r\tau}n(\sigma\sqrt{\tau} - d_1) - Sn(-d_1)] \\ &= e^{-r\tau}N(-d_2) > 0.\end{aligned}\quad (30)$$

4.3 Relationship between call and put option prices and the interest rate (Rho)

Consider the relationship between the price of the call option and the rate of interest, $\partial C/\partial r$:

$$\begin{aligned}\partial C/\partial r &= \tau Ke^{-r\tau}N(d_2) + S(\partial N(d_1)/\partial d_1)(\partial d_1/\partial r) - Ke^{-r\tau}(\partial N(d_2)/\partial d_2)(\partial d_2/\partial r) \\ &= \tau Ke^{-r\tau}N(d_2) + Sn(d_1)(\partial d_1/\partial r) - Ke^{-r\tau}n(d_2)(\partial d_2/\partial r) \\ &= \tau Ke^{-r\tau}N(d_2) + (\partial d_1/\partial r)[Sn(d_1) - Ke^{-r\tau}n(d_1 - \sigma\sqrt{\tau})] \\ &= \tau Ke^{-r\tau}N(d_2) > 0.\end{aligned}\quad (31)$$

Since the right-hand side of equation (31) is positive, this implies that the price of a call

option is positively related to the rate of interest. This comparative static relationship is commonly referred to as the option's *rho*.

Intuition suggests that since the price of a call option is positively related to the rate of interest, one would expect to find that an inverse relationship exists between the price of a put option and the rate of interest, which we confirm in equation (32):

$$\begin{aligned}
\partial P/\partial r &= -\tau K e^{-r\tau} N(-d_2) + K e^{-r\tau} (\partial N(-d_2)/\partial d_2)(\partial d_2/\partial r) + S(\partial N(-d_1)/\partial d_1)(\partial d_1/\partial r) \\
&= -\tau K e^{-r\tau} N(-d_2) + K e^{-r\tau} n(-d_2)(\partial d_2/\partial r) - S n(-d_1)(\partial d_1/\partial r) \\
&= -\tau K e^{-r\tau} N(-d_2) + (\partial d_1/\partial r)[K e^{-r\tau} n(\sigma\sqrt{\tau} - d_1) - S n(-d_1)] \\
&= -\tau K e^{-r\tau} N(-d_2) < 0.
\end{aligned} \tag{32}$$

4.4 Relationship between call and put option prices and the passage of time (Theta)

Consider next the relationship between the price of the call option and the time to expiration, $\partial C/\partial t$ (also known as “theta”):

$$\partial C/\partial t = S(\partial N(d_1)/\partial d_1)(\partial d_1/\partial t) - K e^{-r\tau} (\partial N(d_2)/\partial d_2)(\partial d_2/\partial t) - r K e^{-r\tau} N(d_2). \tag{33}$$

Substituting $K = S e^{-d_1\sigma\sqrt{\tau} + (r + .5\sigma^2)\tau}$ into equation (33) and simplifying further, we obtain

$$\begin{aligned}
\partial C/\partial t &= S[n(d_1)(\partial d_1/\partial t) - e^{-d_1\sigma\sqrt{\tau} + (r + .5\sigma^2)\tau} n(d_1 - \sigma\sqrt{\tau})(\partial d_2/\partial t)] - r K e^{-r\tau} N(d_2) \\
&= S \frac{1}{\sqrt{2\pi}} [(\partial d_1/\partial t) e^{-.5d_1^2} - (\partial d_2/\partial t) e^{-d_1\sigma\sqrt{\tau} + (r + .5\sigma^2)\tau - r\tau - .5(d_1 - \sigma\sqrt{\tau})^2}] - r K e^{-r\tau} N(d_2) \\
&= S \frac{1}{\sqrt{2\pi}} [(\partial d_1/\partial t) e^{-.5d_1^2} - (\partial d_2/\partial t) e^{-d_1\sigma\sqrt{\tau} + d_1\sigma\sqrt{\tau} + r\tau - r\tau + .5\sigma^2\tau - .5\sigma^2\tau - .5d_1^2}] - r K e^{-r\tau} N(d_2) \\
&= S n(d_1)[(\partial d_1/\partial t) - (\partial d_2/\partial t)] - r K e^{-r\tau} N(d_2) \\
&= -S n(d_1) \frac{.5\sigma}{\sqrt{\tau}} - r K e^{-r\tau} N(d_2) < 0.
\end{aligned} \tag{34}$$

Since $\partial C/\partial t < 0$, this implies (other things equal) that as time passes, the call option becomes less valuable. Next, we consider the relationship between the price of a put option and the passage of time. Since we know from put-call parity that $P = C + K e^{-r\tau} - S$, it

follows that

$$\begin{aligned}
\partial P/\partial t &= \partial C/\partial t + rKe^{-r\tau} \\
&= -Sn(d_1)\frac{.5\sigma}{\sqrt{\tau}} - rKe^{-r\tau}N(d_2) + rKe^{-r\tau} \\
&= -Sn(d_1)\frac{.5\sigma}{\sqrt{\tau}} + rKe^{-r\tau}N(-d_2)
\end{aligned} \tag{35}$$

It is not possible to sign $\partial P/\partial t$, because the first term is negative and the second term is positive. Therefore, the relationship between put value and the passage of time is ambiguous.

4.5 Relationship between call and put option prices and volatility of the underlying asset (Vega)

Finally, consider the relationship between the price of the call option and the volatility of the underlying asset, $\partial C/\partial\sigma$:

$$\partial C/\partial\sigma = S\frac{\partial N(d_1)}{\partial d_1}\frac{\partial d_1}{\partial\sigma} - Ke^{-r\tau}\frac{\partial N(d_2)}{\partial d_2}\frac{\partial d_2}{\partial\sigma}. \tag{36}$$

Substituting $\frac{\partial N(d_2)}{\partial d_2} = n(d_2) = n(d_1 - \sigma\sqrt{\tau})$, $\frac{\partial d_2}{\partial\sigma} = \frac{\partial d_1}{\partial\sigma} - \sqrt{\tau}$, and $K = Se^{-d_1\sigma\sqrt{\tau} + (r + .5\sigma^2)\tau}$ into the right-hand side of equation (36) and simplifying further, we obtain

$$\begin{aligned}
\partial C/\partial\sigma &= S\left[n(d_1)\frac{\partial d_1}{\partial\sigma} - e^{-d_1\sigma\sqrt{\tau} + (r + .5\sigma^2)\tau - r\tau}n(d_1 - \sigma\sqrt{\tau})\frac{\partial d_2}{\partial\sigma}\right] \\
&= S\left[\frac{e^{-.5d_1^2}}{\sqrt{2\pi}}\frac{\partial d_1}{\partial\sigma} - \frac{e^{-d_1\sigma\sqrt{\tau} + r\tau + .5\sigma^2\tau - r\tau - .5(d_1 - \sigma\sqrt{\tau})^2}}{\sqrt{2\pi}}\left(\frac{\partial d_1}{\partial\sigma} - \sqrt{\tau}\right)\right].
\end{aligned} \tag{37}$$

Since $e^{-d_1\sigma\sqrt{\tau} + r\tau + .5\sigma^2\tau - r\tau - .5(d_1 - \sigma\sqrt{\tau})^2} = e^{-.5d_1^2}$ and $n(d_1) = \frac{e^{-.5d_1^2}}{\sqrt{2\pi}}$, equation (37) may be rewritten as

$$\begin{aligned}
\partial C/\partial\sigma &= Sn(d_1)\left[\frac{\partial d_1}{\partial\sigma} - \left(\frac{\partial d_1}{\partial\sigma} - \sqrt{\tau}\right)\right] \\
&= Sn(d_1)\sqrt{\tau} > 0.
\end{aligned} \tag{38}$$

As equation (38) indicates, the price of a call option is positively related to the volatility of the underlying asset. This comparative static relationship is commonly referred to as the option's *vega*.

Next, we consider the relationship between the price of a put option and the volatility of the underlying asset, $\partial P/\partial\sigma$:

$$\begin{aligned}\partial P/\partial\sigma &= Ke^{-r\tau}\frac{\partial N(-d_2)}{\partial d_2}\frac{\partial d_2}{\partial\sigma} - S\frac{\partial N(-d_1)}{\partial d_1}\frac{\partial d_1}{\partial\sigma} \\ &= -Ke^{-r\tau}n(-d_2)\frac{\partial d_2}{\partial\sigma} + Sn(-d_1)\frac{\partial d_1}{\partial\sigma}.\end{aligned}\tag{39}$$

Substituting $-d_2 = \sigma\sqrt{\tau} - d_1$, $\frac{\partial d_2}{\partial\sigma} = \frac{\partial d_1}{\partial\sigma} - \sqrt{\tau}$, and $K = Se^{-d_1\sigma\sqrt{\tau}+(r+.5\sigma^2)\tau}$ into the right hand side of equation (39) and simplifying further yields:

$$\begin{aligned}\partial P/\partial\sigma &= S[n(-d_1)\frac{\partial d_1}{\partial\sigma} - \frac{e^{-d_1\sigma\sqrt{\tau}+r\tau+.5\sigma^2\tau-r\tau-.5(\sigma\sqrt{\tau}-d_1)^2}}{\sqrt{2\pi}}(\frac{\partial d_1}{\partial\sigma} - \sqrt{\tau})] \\ &= Sn(-d_1)[\frac{\partial d_1}{\partial\sigma} - (\frac{\partial d_1}{\partial\sigma} - \sqrt{\tau})] \\ &= Sn(-d_1)\sqrt{\tau} > 0.\end{aligned}\tag{40}$$

As equation (40) indicates, the price of a put option is positively related to the volatility of the underlying asset. Table 1 summarizes the comparative statics of the Black-Scholes call and put option pricing formulas.

Table 1. Comparative Statics of the Black-Scholes Model

Derivative	Call Option	Put Option
$\frac{\partial C}{\partial S}$ and $\frac{\partial P}{\partial S}$ (delta)	$\frac{\partial C}{\partial S} = N(d_1) > 0$	$\frac{\partial P}{\partial S} = -N(-d_1) < 0$
$\frac{\partial C}{\partial K}$ and $\frac{\partial P}{\partial K}$	$\frac{\partial C}{\partial K} = -e^{-r\tau} N(d_2) < 0$	$\frac{\partial P}{\partial K} = e^{-r\tau} N(-d_2) > 0$
$\frac{\partial C}{\partial r}$ and $\frac{\partial P}{\partial r}$ (rho)	$\frac{\partial C}{\partial r} = \tau K e^{-r\tau} N(d_2) > 0$	$\frac{\partial P}{\partial r} = -\tau K e^{-r\tau} N(-d_2) < 0$
$\frac{\partial C}{\partial t}$ and $\frac{\partial P}{\partial t}$ (theta)	$\frac{\partial C}{\partial t} = -S n(d_1) \frac{.5\sigma}{\sqrt{\tau}} - r K e^{-r\tau} N(d_2) < 0$	$\frac{\partial P}{\partial t} = -S n(d_1) \frac{.5\sigma}{\sqrt{\tau}} + r K e^{-r\tau} N(-d_2) \stackrel{?}{>} 0$
$\frac{\partial C}{\partial \sigma}$ and $\frac{\partial P}{\partial \sigma}$ (vega)	$\frac{\partial C}{\partial \sigma} = S n(d_1) \sqrt{\tau} > 0$	$\frac{\partial P}{\partial \sigma} = S n(-d_1) \sqrt{\tau} > 0$

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