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CML TO SML: AN ALTERNATIVE APPROACH

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INTRODUCTION

Numerous techniques exist for demonstrating the logical relationship that exists between the Capital Market Line (CML) and the Security Market Line (SML). These techniques typically range from fairly rigorous calculus-based approaches (e.g., Fama, 1976) to simpler, more intuitive graphical approaches (e.g., Spence, 1984). This paper develops an alternative approach which is based in its entirety upon the mathematics of the expectations operator. We show that, given the CML, three logical corollaries naturally follow, the third of which is the SML equation itself. It is assumed that the reader understands the basic elements of probability theory and capital market theory, including (1) how to calculate expected returns, standard deviations, covariances, and correlations from ex ante data, and (2) the logical relationship between the efficient frontier comprised solely of risky assets and the efficient frontier comprised of riskless as well as risky assets (which is, of course, the CML).

The paper is organized as follows. In the next section, the notation is presented which will be used throughout the paper. The final section concludes with the alternative approach to the derivation of the CAPM.

NOTATION

R_f = the rate of return on a riskless security;

$E(\cdot)$ = expectation operator;

$E(R_i)$ = expected return on security i;

σ_i^2 = $E\{[R_i - E(R_i)]^2\}$
= variance of returns on security i;

σ_i = standard deviation of returns on security i;

σ_{ij} = $E\{[R_i - E(R_i)][R_j - E(R_j)]\}$
= covariance between the returns on securities i and j;

ρ_{ij} = $\sigma_{ij}/(\sigma_i\sigma_j)$
= correlation between the returns on securities i and j;

β_i = $\sigma_{im}/\sigma_m^2 = (\sigma_i/\sigma_m)\rho_{im}$
= 'beta' of the ith security;

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X_i = proportion of initial wealth invested in security i;
X_f = proportion of initial wealth invested in the riskless asset;
X_m = proportion of initial wealth invested in the market portfolio;

$$E(R_p) = \sum_{i=1}^{n} X_i E(R_i)$$

= expected return on an n-asset portfolio;

$$\sigma_p^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} X_i X_j \sigma_{ij}$$

= variance of returns on an n-asset portfolio;

$$\beta_p = \sum_{i=1}^{n} X_i \beta_i$$

= 'beta' of an n-asset portfolio.

AN ALTERNATIVE APPROACH TO THE DERIVATION OF THE CAPM

Theorem (Capital Market Line) The efficient set of portfolios is comprised of all possible linear combinations of the riskless security and the market portfolio. The equation for the line which describes the expected returns on these portfolios is given by

$$E(R_p) = R_f + \left(\frac{E(R_m) - R_f}{\sigma_m}\right) \sigma_p. \tag{1}$$

Proof See Fama (1976), chapter 8.

Corollary No. 1 The returns on efficient portfolios are perfectly positively correlated with the return on the market portfolio.

Proof By definition, $\rho_{pm} \equiv \sigma_{pm}/(\sigma_p \sigma_m)$, $E(R_p) \equiv X_f R_f + X_m E(R_m)$, $\sigma_p \equiv X_m \sigma_m$, and $\sigma_{pm} \equiv E\{[R_p - E(R_p)][R_m - E(R_m)]\}$. With these relationships in mind, we will write out the formula for σ_{pm}:

$$\begin{aligned} \sigma_{pm} &\equiv E\{[R_p - E(R_p)][R_m - E(R_m)]\} \\ &= E\{[X_f R_f + X_m R_m - X_f R_f - X_m E(R_m)][R_m - E(R_m)]\} \\ &= E\{X_m [R_m - E(R_m)]^2\} \\ &= X_m \sigma_m^2. \end{aligned} \tag{2}$$

Therefore,

$$\rho_{pm} = \sigma_{pm}/(\sigma_p \sigma_m) = X_m \sigma_m^2/(X_m \sigma_m \sigma_m) = 1 \tag{3}$$

Q.E.D.

Corollary No. 2 Efficient portfolios have no unsystematic risk; viz., 100 percent of the riskiness of any arbitrary efficient portfolio is that which is inherent in the market portfolio.

Proof We can write the return on any portfolio p in the following manner:

$$R_p = \alpha_p + \beta_p R_m + \epsilon_p \tag{4}$$

where α_p = a constant = $E(R_p) - \beta_p E(R_m)$;
β_p = slope coefficient = $\sigma_{pm}/\sigma_m^2 = (\sigma_p/\sigma_m)\rho_{pm}$;
$E(\epsilon_p) = 0$;
$E(\epsilon_p^2) = \sigma_\epsilon^2$;
$E(R_m \epsilon_p) = 0$.

Next, we can write the variance of R_p as follows:

$$\begin{aligned}
\sigma_p^2 &\equiv E\{[R_p - E(R_p)]^2\} \\
&= E\{[\alpha_p + \beta_p R_m + \epsilon_p - (\alpha_p + \beta_p R_m)]^2\} \\
&= E\{[\beta_p[R_m - E(R_m)] + \epsilon_p]^2\} \\
&= E\{\beta_p^2[R_m - E(R_m)]^2 + \epsilon_p^2 + 2\beta_p[R_m - E(R_m)]\epsilon_p\} \\
&= \beta_p^2 E[R_m - E(R_m)]^2 + E(\epsilon_p^2) \\
&= \beta_p^2\sigma_m^2 + \sigma_\epsilon^2.
\end{aligned} \tag{5}$$

Equation (5) can be given what is intuitively a rather appealing interpretation; viz., the total risk inherent in the returns on any given portfolio of securities is equal to the sum of systematic risk (i.e., $\beta_p^2\sigma_m^2$) and unsystematic risk (i.e., σ_ϵ^2). Dividing both sides of equation (5) through by σ_p^2 yields equation (6):

$$1 = (\beta_p^2\sigma_m^2)/\sigma_p^2 + \sigma_\epsilon^2/\sigma_p^2. \tag{6}$$

The intuition underlying equation (6) is also rather appealing. The proportion of total risk which is accounted for by its systematic component is given by the expression $(\beta_p^2\sigma_m^2)/\sigma_p^2$, while the proportion of total risk accounted for by its unsystematic component is given by $\sigma_\epsilon^2/\sigma_p^2$.

Next, since $\beta_p = (\sigma_p/\sigma_m)\rho_{pm}$, we can rewrite $(\beta_p^2\sigma_m^2)/\sigma_p^2 = \rho_{pm}^2$. Of course, this result is to be expected since regression theory tells us that the ratio $(\beta_p^2\sigma_m^2)/\sigma_p^2$ is equal to R^2, or the coefficient of determination, which is itself equal to the square of the correlation coefficient. Since we have already demonstrated in the first corollary that the returns on all efficient portfolios are perfectly positively correlated with the returns on the market portfolio, this means that $(\beta_p^2\sigma_m^2)/\sigma_p^2$ must assume a value of unity for such portfolios. Consequently, it follows from equation (6) that $\sigma_\epsilon^2/\sigma_p^2$ is zero, which in turn implies that efficient portfolios have no unsystematic risk. *Q.E.D.*

Corollary No. 3 (Security Market Line) Given the CML, it follows that the equilibrium expected return on individual securities is given by the Security Market Line (SML) equation:

$$E(R_i) = R_f + [E(R_m) - R_f]\beta_i. \tag{7}$$

Proof Since $\sigma_p^2 = \beta_p^2\sigma_m^2$ for efficient portfolios, we can write $\beta_p = \sigma_p/\sigma_m$. Substituting this result into equation (1), we obtain

$$E(R_p) = R_f + [E(R_m) - R_f]\beta_p. \tag{8}$$

Therefore, risk premiums expected from investing in efficient portfolios can be expressed either as (1) the product of the market price of risk ($[E(R_m) - R_f]/\sigma_m$) and the portfolio's standard deviation (σ_p) (cf. equation (1)), or (2) the product of the market risk premium ($[E(R_m) - R_f]$) and the portfolio's beta (β_p) (cf. equation (8)). These two expressions are identical to each other because efficient portfolios have no unsystematic risk.

Next, since

$$E(R_p) \equiv \sum_{i=1}^{n} X_i E(R_i), \tag{9a}$$

$$\beta_p \equiv \sum_{i=1}^{n} X_i \beta_i, \quad \text{and} \tag{9b}$$

$$\sum_{i=1}^{n} X_i \equiv 1, \tag{9c}$$

equation (8) can be rewritten as

$$\sum_{i=1}^{n} X_i E(R_i) = \sum_{i=1}^{n} X_i [R_f + [E(R_m) - R_f]\beta_i]. \tag{10}$$

Having established in equation (8) that the SML equation holds for efficient portfolios, equation (10) demonstrates that the SML equation must also necessarily hold for each of the individual securities which comprise such portfolios. This result obtains due to the identities given in equations (9a)–(9c).

While the above result may appear to be somewhat tautological, the logic underlying it is perfectly consistent with the basic economic notion of arbitrage. To show this, suppose that equation (8) holds for efficient portfolios while equation (10) does not hold for one or more of the individual assets which comprise such portfolios. If this were the case, then investors would be able to generate excess trading profits by adopting appropriate arbitrage strategies. For example, investors could fund investments in correctly priced (underpriced) securities with the proceeds derived from short selling overpriced (correctly priced) securities. Obviously, such profit opportunities cannot possibly exist in a portfolio equilibrium. *Q.E.D.*

REFERENCES

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