Teaching the Economics and Convergence of the Binomial and Black-Scholes Option Pricing Formulas

James R. Garven* and James I. Hilliard†

December 2018

Abstract

The Black-Scholes option pricing model is a particularly important part of the finance curriculum. Unfortunately, undergraduate students often do not understand this model particularly well since the math upon which the model is based lies well beyond the scope of standard finance textbooks. This paper accomplishes two important tasks: 1) it captures the most important economic intuitions underlying Black-Scholes with a simple single-period binomial model, and 2) it provides a spreadsheet model demonstrating how binomial model probabilities and prices numerically converge to their Black-Scholes counterparts. Recommendations for teaching option pricing and convergence are provided by a hypothetical managerial compensation problem.

Keywords: delta hedging, portfolio replication, risk neutral valuation, convergence

1 Introduction

We have found that undergraduate finance students often do not comprehend the logical connections which exist between the binomial and Black-Scholes option pricing models. This is unfortunate since these models represent particularly important aspects of the finance curriculum. Besides their uses in investment courses, these models also provide particularly valuable analytic tools for studying corporate finance topics as real options, agency theory, risk management, credit risk, etc.

*Professor of Finance & Insurance and Frank S. Groner Memorial Chair of Finance, Hankamer School of Business, Baylor University, Foster 320.39, One Bear Place #98004, Waco, TX 76798, telephone: 254-710-6207, Email: James_Garven@baylor.edu

†Associate Professor of Finance and Arndt Professor, W. A. Franke College of Business, Northern Arizona University, P.O. Box 15066, Flagstaff, AZ 86011, telephone: 928-523-7901, e-mail: james.hilliard@nau.edu

In this paper, we provide a pedagogical framework which not only introduces the basic concepts necessary to understand option valuation principles, but also illustrates the convergence from the binomial model to the Black-Scholes model. In the process, we provide suggestions for walking students through the mathematical portions, as well as a case study example in the form of a managerial compensation example. Finally, we provide a spreadsheet template which numerically and graphically illustrates the convergence without relying on macros or other more challenging coding techniques beyond basic features of Excel.

To motivate class discussion, we suggest an option – the executive stock option. Specifically, we consider a senior manager's choice between an option grant for 1,000 shares of company stock, expiring in one year with a \$60 per share exercise price, and a bonus with a present value of \$3,000. The stock trades for \$50, and in our initial numerical example, the price will either rise to \$62.50 or fall to \$40 one year from today. The challenge for the manager is to determine how the value of this option grant compares with the value of the bonus which has been offered. In subsequent iterations of this example, the binomial outcomes suggested here are replaced with outcomes based on the volatility of the company's stock, and the expiration date is extended beyond one year.

In the next section of this paper, we showcase the single-period versions of the delta hedging and replicating portfolio approaches to pricing options, and show how both of these methods encompass the risk neutral valuation approach. All three methods rely upon the so-called "no-arbitrage" principle, where arbitrage refers to the opportunity to earn riskless profits by taking advantage of price differences between virtually identical investments; i.e., arbitrage represents the financial equivalent of a "free lunch". However, since competition ensures that opportunities to make riskless profits are quickly dissipated, so-called "arbitrage-free" prices for options emerge. While leading financial derivatives textbooks by Hull (2015) and McDonald (2013) also emphasize risk neutral valuation, Hull (pp. 274-280) motivates risk neutral valuation via the delta hedging approach, whereas McDonald (pp. 293-300) motivates risk neutral valuation via the replicating portfolio approach. In Section 2, we motivate risk neutral valuation via both of these methods and formally show that the delta hedging and replicating portfolio approaches constitute sufficient conditions for risk neutral valuation. In Section 3, we expand the risk neutral valuation model to two or more periods, and show how it generalizes as the Cox, Ross, and Rubinstein (1979) binomial option pricing formula. In Section 4, we illustrate how probabilities and prices under the Cox-Ross-Rubinstein model numerically converge to Black-Scholes probabilities and prices. We assume that options are European; i.e., exercise may only occur on the expiration date. Furthermore, the underlying asset pays off only at the option's expiration date; i.e., no cash flows occur prior to the expiration of the option contract. Concluding remarks are provided in the fifth section of the paper.

2 The Single-Period Model

2.1 Delta Hedging Approach

Suppose the manager initially implements the delta hedging approach for the purpose of determining the "arbitrage-free" value of the option grant which has been offered. The current price per share of the underlying company stock is S, and one timestep (δt) from now, the stock will assume one of the following two values: $S_u = uS$ or $S_d = dS$, where $u > 1$ and $d < 1$. Initially, we assume that S = \$50, $u = 1.25$, $d = .8$, $\delta t = 1$ (one year), the exercise price K = \$60, and the continuously compounded riskless rate of interest r = 3%. Figure 1 shows the binomial "tree" for the current (known) stock price and also the future (state-contingent) stock prices, and Figure 2 shows the binomial tree for the current (unknown) call option price and also the future (state-contingent) call option prices.

Next, the manager forms a "hedge" portfolio consisting of a long position in one call option and a short position in Δ shares of stock. This portfolio is called a hedge portfolio because movements in the value of the short stock position hedge, or offset the effect of movements in the value of the long call option position. The current market value of this hedge portfolio is

$$V_H = C - \Delta S = C - \Delta 50. \tag{1}$$

At the up (u) node, the value of the hedge portfolio is equal to $V_H^u = C_u - \Delta S_u = 2.50 - \Delta 62.50$, and at the down ($d$) node, the value of the hedge portfolio is equal to $V_H^d = C_d - \Delta S_d = 0 - \Delta 40$. Suppose we solve for Δ such that the hedge portfolio is riskless; i.e., $V_H^u = V_H^d$. Since $V_H^u = V_H^d$, this implies that $2.50 - \Delta 62.50 = -\Delta 40$ and $\Delta = 0.111$. Substituting $\Delta = 0.111$ back into the expressions for V_H^u and V_H^d, we obtain $V_H^u = V_H^d = -\$4.44$. An example of this solution for whiteboard or presentation slide explanation is provided in Figure 3. Thus, the terminal value of a riskless hedge portfolio comprised of one call option and a short position in one-ninth of a share of stock is equivalent in value to a *short* position in a "synthetic" riskless bond worth \$4.44 one year from now, and the present value of this short bond position is $V_H = -4.44e^{-.03} = -\$4.31$.

Even though the call option and the stock have completely different cash flow characteristics than a riskless bond, the riskless hedge portfolio consisting of these two securities creates a "synthetic" riskless bond in the sense that its cash flows artificially mimic the riskless bond cash flows. Under no-arbitrage conditions, the price of the synthetic bond must equal the price of the actual bond with the same payoffs. So, for a given stock value, the price of the call option which satisfies this no-arbitrage condition is the arbitrage-free price. Since $V_H = C - (0.111)\,50$, this implies that the arbitrage-free price for the call option is $C = \$1.24$, which in turn implies that the option grant is worth 1,000 x \$1.24, or \$1,240. Since the value of the bonus exceeds the value of the option grant, our manager will rationally prefer the bonus.

While we would not expect a manager to be offered a put option grant, it is nevertheless worthwhile to consider how to price an otherwise identical put option with an exercise price of \$60. Since the arbitrage-free price for the call option is \$1.24, we rely upon the put-call parity equation (Stoll (1969)) to determine the arbitrage-free price of an otherwise identical put option.

The put-call parity equation is given by equation (2):

$$C + Ke^{-r\delta t} = P + S. \tag{2}$$

Thus,

$$P = C + Ke^{-r\delta t} - S = \$1.24 + 60e^{-.03} - \$50 = \$9.47. \tag{3}$$

We can also determine the arbitrage-free price for the put option via the delta hedging approach. Since the state-contingent prices of a put option and its underlying stock are inversely related, we form a hedge portfolio consisting of a long position in one put option and a long position in Δ shares of stock. The current value of this portfolio is

$$V_H = P + \Delta S = P + \Delta 50. \tag{4}$$

At node u, the value of the hedge portfolio is equal to $V_H^u = P_u + \Delta S_u = Max\,(K - 62.50, 0) + \Delta 62.50 = 0 + \Delta 62.50$, and at node d, the value of the hedge portfolio is equal to $V_H^d = P_d + \Delta S_d = Max\,(K - 40, 0) + \Delta 40 = 20 + \Delta 40$. Suppose we select Δ such that the hedge portfolio is riskless; i.e., $V_H^u = V_H^d$ implies that $\Delta 62.50 = 20 + \Delta 40$; thus $\Delta = 0.889$. Substituting $\Delta = 0.889$ back into the expressions for V_H^u and V_H^d, we obtain $V_H^u = V_H^d = 55.56$. These calculations could be illustrated on the whiteboard or in a presentation slide in a similar manner to Figure 3. Thus, the terminal value of a riskless hedge portfolio comprised of one put option and a long position in a eight-tenths of a share of stock is equivalent in value to a *long* position in a synthetic riskless bond worth \$55.56 one year from now. The present value of this long bond position is $V_H = \$55.56e^{-.03} = \$53.91$, which implies that $P = \$9.47$.

In the next section, we explore an alternative approach to option valuation. Rather than infer the value of an option by pricing a synthetic riskless bond, we infer option value from synthetic options which are created by replicating the state-contingent payoffs on actual options with combinations of the underlying stock and a riskless bond.

2.2 Replicating Portfolio Approach

Another way for the manager to evaluate the value of the option grant is to create a replicating portfolio. Under this trading strategy, the manager replicates the call option payoffs at nodes u and d by purchasing Δ shares of stock today and financing part of this investment by borrowing money. The current market value of the replicating portfolio must equal the current market value of the option; if the replicating portfolio and the option have different market values, the manager can earn positive profits with zero risk and zero net investment by buying the less expensive investment and shorting the more expensive one. Thus, we invoke the no-arbitrage condition to establish that the arbitrage-free price of the call option must equal the value of its replicating portfolio.

To replicate the payoffs of the call option, the manager forms a hypothetical portfolio consisting of Δ shares of stock and $\$B$ in riskless bonds. The initial cost of forming such a portfolio is $\$\left(\Delta S + B\right)$. When the option expires, the value will be determined by either equation (5) or (6), depending on the end state:

$$C_u = \Delta u S + e^{r\delta t} B, \text{and} \tag{5}$$

$$C_d = \Delta d S + e^{r\delta t} B. \tag{6}$$

Note that the first term in equation (5) represents the anticipated value of the underlying stock at node u (uS) multiplied by the number (or fraction) of shares held in the underlying stock. The second term represents the future value of the bond, assuming continuous compounding at the annual rate of r during the δt time interval. Equation (6) provides the corresponding value of the replicating portfolio at node d. The manager will determine how many shares to purchase, and how much to borrow by simultaneously solving equations (5) and (6) for Δ and B, obtaining:

$$\Delta = \frac{C_u - C_d}{S\left(u - d\right)} \geq 0, \text{and} \tag{7}$$

$$B = \frac{uC_d - dC_u}{e^{r\delta t}\left(u-d\right)} \leq 0. \tag{8}$$

Note that the equalities in equations (7) and (8) only hold when $C_u = C_d = 0$; i.e., only if the call option always expires out of the money. Otherwise, $\Delta > 0$ and $B < 0$; i.e., node u and d call option payoffs correspond to payoffs at these same nodes on a margined investment in the stock based on the Δ and B values obtained from equations (7) and (8).

Next, let's reconsider these equations in light of our numerical example. From equations (7) and (8), $\Delta = \frac{C_u - C_d}{S\left(u-d\right)} = .111$ and $B = \frac{uC_d - dC_u}{e^{r\delta t}\left(u-d\right)} = \frac{1.25\left(0\right) - .8\left(2.5\right)}{e^{.03}\left(.45\right)} = -4.31$. Note that Δ here is the same as the Δ calculated under the delta hedging approach, and the value of B is the same as the value of V_H in the earlier approach. The manipulation of these equations can be worked out on the whiteboard or presentation slides as shown in Figure 4. Thus, the manager can replicate the call option by purchasing one-ninth of a share of stock for \$5.55 and borrowing \$4.31. Since the value of the replicating portfolio is $\$\left(\Delta S + B\right)=$ \$5.55 - 4.31 = \$1.24, this must also be the arbitrage-free value of the call option. Therefore, the decision regarding the choice between the option grant or the bonus is the same as in the previous section; since the bonus is worth more than option grant (\$3,000 compared with \$1,240), our manager will rationally prefer the bonus.

Following similar logic, we can determine the value of the replicating portfolio for the put option. Suppose we form a portfolio consisting of Δ shares of stock and $\$B$ in riskless bonds. The initial cost of forming such a portfolio is $\$\left(\Delta S + B\right)$. At expiration,

$$P_u = \Delta uS + e^{r\delta t}B, \text{and} \tag{9}$$

$$P_d = \Delta dS + e^{r\delta t}B. \tag{10}$$

Simultaneously solving equations (9) and (10) for Δ and B, we obtain:

$$\Delta = \frac{P_u - P_d}{S\left(u-d\right)} \leq 0, \text{and} \tag{11}$$

$$B = \frac{uP_d - dP_u}{e^{r\delta t}(u-d)} \geq 0. \tag{12}$$

Note that the equalities in equations (11) and (12) only hold when $P_u = P_d = 0$; i.e., only if the put option always expires out of the money. Otherwise, $\Delta < 0$ and $B > 0$; i.e., put option payoffs correspond to payoffs at these same nodes on an investment consisting of a short position in the stock, coupled with a long position in a riskless bond based on Δ and B values obtained from equations (11) and (12).

Next, let's reconsider these equations in light of our numerical example. From equations (11) and (12), $\Delta = \frac{P_u - P_d}{S(u-d)} = -20/22.50 = -.889$ and $B = \frac{uP_d - dP_u}{e^{r\delta t}(u-d)} = \frac{1.25(20) - .8(0)}{e^{.03}(.45)} = \$53.91$. Thus, we can replicate the put option by shorting eight-ninths of a share for \$44.44 and lending \$53.91. Since the value of the replicating portfolio is $\$(\Delta S + B) = -\$44.44 + \$53.91 = \$9.47$, this must also be the arbitrage-free price of the put option.

Even though the delta hedging and replicating portfolio approaches to option valuation are motivated differently, both approaches yield the same arbitrage-free prices for call and put options. Furthermore, neither the delta hedging approach of section 2.1 nor the replicating portfolio approach of section 2.2 require the use of probabilities for calculating option prices. This is a somewhat counter-intuitive result, since after all, one would think that the value of an option *should* depend, at least in part, upon the probabilities of up and down movements in the price of the underlying stock. This insight is important as we move forward with one more example of a binomial pricing model approach which relies upon risk neutral, or risk-adjusted probabilities to calculate arbitrage-free option prices. As we show next, this approach is implied by both the delta hedging and risk neutral valuation approaches.

2.3 Risk Neutral Valuation Approach

Next, we consider the risk neutral valuation approach to pricing options. This approach is popular due to its simplicity. However, the most challenging aspect of this approach involves understanding the how risk neutral probabilities are determined, and what they mean in practice.

In sections 2.1 and 2.2, we inferred arbitrage-free prices for call and put options by either creating a synthetic riskless bond (via the delta hedging approach) or by creating synthetic call and put options (via the replicating portfolio approach). Investor risk preferences are not a factor in the formation of arbitrage-free prices because risk is eliminated under both of these trading strategies. Arbitrage-free prices obtain so long as investors are motivated (due to a preference for more versus less wealth) to take advantage of opportunities to earn riskless arbitrage profits. Therefore, since the valuation relationship between an option and its underlying asset does not depend upon investor risk preferences, it follows that options may be priced *as if* investors are *risk neutral*. This idea is a foundational principle for the risk neutral valuation approach.

We begin our analysis by showing the relationship which exists between the expected return on the underlying stock (μ) and the probability of an up move (p). Note that

$$E\left(S_{\delta t}\right) = puS + (1-p)\,dS = e^{\mu \delta t} S, \tag{13}$$

where $E\left(S_{\delta t}\right)$ represents the expected stock price at expiration and μ corresponds to the annualized expected return on the stock. Solving equation (13) for p, we find that

$$p = \frac{\left(e^{\mu \delta t} - d\right)}{(u-d)}. \tag{14}$$

A whiteboard or presentation slide example for solving for μ from equation (14) is presented in Figure 5.

Suppose that investors are *risk averse* and that $p = 0.60$. Solving equation (14) for μ, we find that $\mu = \frac{\ln\left(pu + (1-p)\,d\right)}{\delta t} = \frac{\ln\left(.6\,(1.25) + (.4)\,.8\right)}{1} = 6.77\%$. Given these probabilities and payoffs, risk averse investors demand an (annualized) expected rate of return on the underlying stock which is 3.77 percentage points higher than the riskless rate of interest.

But suppose instead that investors are *risk neutral*. In a risk neutral market, the expected return on a risky asset is the same as the expected return on a safe asset, because risk neutral investors do not demand a risk premium. This implies that the expected stock price one period from now in such a market is written as follows:

$$\hat{E}\left(S_{\delta t}\right) = quS + (1-q)\, dS = e^{r\delta t} S, \tag{15}$$

where $\hat{E}\left(S_{\delta t}\right)$ corresponds to the risk neutral expected value of the stock one period from now and q corresponds to the risk neutral probability of an up move. Solving equation (15) for q, we find that

$$q = \frac{e^{r\delta t} - d}{(u-d)} = \frac{e^{.03} - .8}{(.45)} = .5121. \tag{16}$$

Since we know the risk neutral probability q, we can calculate the risk neutral expected values of the call and put option payoffs at expiration by simply weighting these payoffs by their corresponding risk neutral probabilities:

$$\hat{E}\left(C_{\delta t}\right) = qC_u + (1-q)\, C_d, \text{and} \tag{17}$$

$$\hat{E}\left(P_{\delta t}\right) = qP_u + (1-q)\, P_d, \tag{18}$$

where $\hat{E}\left(\cdot\right)$ corresponds to the risk neutral expected value operator. Essentially, $\hat{E}\left(C_{\delta t}\right)$ and $\hat{E}\left(P_{\delta t}\right)$ represent the certainty-equivalent values for the call and put option payoffs at the expiration date. As such, we can determine the prices of these options by simply discounting these certainty-equivalent values for one year at the riskless rate of interest. Thus, the prices for (single-period) European call and put options are given by equations (19) and (20):

$$C = e^{-r\delta t}\hat{E}\left(C_{\delta t}\right) = e^{-r\delta t}\left[qC_u + (1-q)\, C_d\right] = e^{-.03}\left[.5121\,(5)\right] = \$1.24, \text{and} \tag{19}$$

$$P = e^{-r\delta t}\hat{E}\left(P_{\delta t}\right) = e^{-r\delta t}\left[qP_u + (1-q)\, P_d\right] = e^{-.03}\left[.4879\,(20)\right] = \$9.47. \tag{20}$$

Since the risk neutral valuation approach is implied by both the delta hedging and replicating portfolio approaches, the option prices obtained under risk neutral valuation are (not surprisingly) the same arbitrage-free prices as determined under the delta hedging and replicating portfolio approaches. Consequently, the decision regarding the choice between the option grant or the bonus remains the same as in sections 2.1 and 2.2; specifically, since the bonus is worth more than option grant ($3,000 compared with $1,240), our manager will rationally prefer the bonus.

2.4 Risk Neutral Valuation and the Delta Hedging Approach

The manager is now somewhat perplexed and wishes to better understand the logical connections that exist between the delta hedging and replicating portfolio approaches to pricing options and the risk neutral valuation approach. In this section, we show how risk neutral valuation is implied by the delta hedging approach, and in the section following, this is shown for the replicating portfolio approach.

In section 2.1, we formed a hedge portfolio consisting of a long position in one call option and a short position in Δ shares of stock. At the beginning of the binomial tree, the value of the hedge portfolio (as indicated by equation (1)) is $V_H = C - \Delta S$. Since equation (7) indicates that $\Delta = \frac{C_u - C_d}{S(u-d)}$, it follows that

$$V_H = C - \frac{C_u - C_d}{S(u-d)} S = C - \frac{C_u - C_d}{(u-d)}. \tag{21}$$

At expiration, the value of the hedge portfolio will be the same, irrespective of whether the stock moves up or down; i.e., $V_H^u = V_H^d$ implies that $C_u - \frac{C_u - C_d}{(u-d)} u = C_d - \frac{C_u - C_d}{(u-d)} d$. Thus, the arbitrage-free value of the hedge portfolio, V_H, corresponds to the present value of either V_H^u or V_H^d (let's go with V_H^u); i.e., $V_H = C - \frac{C_u - C_d}{(u-d)} = e^{-r\delta t}\left[C_u - \frac{C_u - C_d}{(u-d)} u\right]$ implies that $C = \frac{C_u - C_d}{(u-d)} + e^{-r\delta t}\left[C_u - \frac{C_u - C_d}{(u-d)} u\right]$. Solving for the arbitrage-free price of the call option, we find that

$$
\begin{aligned}
C &= \frac{C_u - C_d + \left[(u-d)\,C_u - uC_u + uC_d\right]e^{-r\delta t}}{u-d} \\
&= \frac{C_u - C_d - dC_u e^{-r\delta t} + uC_d e^{-r\delta t}}{u-d} \\
&= e^{-r\delta t}\left[\frac{e^{r\delta t}-d}{u-d}C_u + \frac{u-e^{r\delta t}}{u-d}C_d\right] \\
&= e^{-r\delta t}\left[qC_u + (1-q)\,C_d\right] \qquad (22)
\end{aligned}
$$

The risk neutral valuation relationship shown in equation (22) is identical to the risk neutral valuation relationship shown in equation (19). Thus, the delta hedging approach implies that a risk neutral valuation relationship exists between a call option and its underlying stock. By symmetry, the analysis shown here also validates that a risk neutral valuation relationship exists between a put option and its underlying stock (cf. equation (20)).

2.5 Risk Neutral Valuation and the Replicating Portfolio Approach

Next, we show how risk neutral valuation is implied by the replicating portfolio approach. As shown in section 2.2, the value of a replicating portfolio $V_{RP} = \Delta S + B$, where $\Delta = \dfrac{C_u - C_d}{S\,(u-d)}$ and $B = \dfrac{uC_d - dC_u}{e^{r\delta t}\,(u-d)}$ (cf. equations (7) and (8)). Thus,

$$
\begin{aligned}
C &= \frac{C_u - C_d}{S\,(u-d)}S + \frac{uC_d - dC_u}{e^{r\delta t}\,(u-d)} \\
&= \frac{e^{r\delta t}\,(C_u - C_d) + uC_d - dC_u}{e^{r\delta t}\,(u-d)} \\
&= e^{-r\delta t}\frac{C_u\left(e^{r\delta t}-d\right) + C_d\left(u-e^{r\delta t}\right)}{(u-d)}. \qquad (23)
\end{aligned}
$$

Since $q = \dfrac{e^{r\delta t}-d}{u-d}$ and $1-q = \dfrac{u-e^{r\delta t}}{u-d}$, substituting q and $1-q$ into the right-hand side of equation (23) yields:

$$C = e^{-r\delta t}\left[qC_u + (1-q)\,C_d\right]. \tag{24}$$

Thus, the replicating portfolio approach implies that a risk neutral valuation relationship exists between a call option and its underlying stock. By symmetry, the analysis shown here also validates that a risk neutral valuation relationship also exists between a put option and its underlying stock (cf. equation (20)).

Now that the logical coherence of the risk neutral valuation, delta hedging, and replicating portfolio approaches to pricing options in a single-period framework has been analytically and numerically demonstrated, our next task involves expanding the risk neutral valuation model to incorporate multiple periods.

3 The Multi-Period Model

In the previous section of the paper, we assumed that the manager's option grant expired after one one-year period. In this section, we expand the model to allow for multiple periods prior to expiration. Specifically, we will expand the risk neutral valuation model to two or more periods, and then show how it generalizes as the Cox-Ross-Rubinstein binomial option pricing formula.

Suppose that the manager now wishes to determine the value of an otherwise identical option grant for 1,000 shares of company stock, expiring after *two* one-year periods. Figure 6 shows the binomial tree for the current and future stock prices at the up (u), down (d), up-up (uu), up-down (ud), and down-down (dd) nodes, whereas Figure 7 shows the binomial tree for the current and future call option prices at nodes u, d, uu, ud, and dd. The manager will begin at the terminal (uu , ud , and dd) nodes shown in Figure 7, and apply the risk neutral valuation formula given by equation (19) to determine arbitrage-free prices for C_u, C_d, and C. This solution procedure is known as "backward induction" since it requires working backwards from the terminal state-contingent values of the call option to the present.

In Figure 7, since the stock only finishes in-the-money at the uu node, it follows that $C_{uu} = \$78.13 - \$60 = \$18.13$, whereas $C_{ud} = C_{dd} = \$0$. Thus, the arbitrage-free call option price at node u (applying the node u version of equation (19)), is $C_u = e^{-r\delta t}\left[qC_{uu} + (1-q)\,C_{ud}\right]$ $= e^{-.03}\left[.5121\,(\$18.13)\right] = \$9.01$. Since $C_{ud} = C_{dd} = \$0$, it also follows that $C_d = \$0$. Applying equation (19) once again, the manager determines that the current arbitrage-free price of the call option is $C = e^{-r\delta t}\left[qC_u + (1-q)\,C_d\right] = e^{-.03}\left[.5121\,(\$9.01)\right] = \$4.48$. Note that the two-period price is more than three times the single-period price of \$1.24. It is well-known that the value of a call option increases as the time to maturity increases. This results from the fact that the underlying asset has more time to increase in value, thus increasing the value of the option if it expires in-the-money. Returning to our executive compensation example, we can see that an otherwise identical option grant maturing in two years rather than one year is now worth \$4,480, making the option grant more appealing than the bonus.

Although backward induction is required to price the option grant via under the delta hedging and replicating portfolio approaches, it is not necessary under risk neutral valuation. Since the manager's option grant is European and may only be exercised at expiration, intermediate node prices for the option (such as C_u and C_d) are not needed in order to find the current arbitrage-free option price (C), since the value for C depends *solely* on the anticipated terminal values of the option. Therefore, the manager only needs to undertake the following three steps: 1) calculate the risk neutral probability for each node at the expiration date, 2) calculate the risk neutral expected value of the option at expiration, and 3) discount the risk neutral expected value to present value at the riskless rate of interest for the number of periods to expiration.

The valuation of a multi-period option value (with a small number of periods) is quite simple for most students. However, understanding that process requires the building blocks shown above (including the delta hedging and replicating portfolio approaches). Once the multi-period risk neutral valuation model is fully grasped by students, the next step is to introduce them to the "simplified" Cox, Ross, and Rubinstein (1979) approach to pricing options.

The complexity of analysis grows with each additional timestep. Fortunately, Cox, Ross, and Rubinstein (CRR) come to our aid, with their multi-period call option pricing formula as given by equation (25):

$$C = e^{-rT} \sum_{j=0}^{n} \begin{pmatrix} n \\ j \end{pmatrix} q^j (1-q)^{n-j} C_j \tag{25}$$

where $\binom{n}{j} = \frac{n!}{j!\,(n-j)!}$ and T corresponds to a fixed expiration date of T periods from now. Furthermore, n corresponds to the total number of timesteps between now and date T, δt corresponds to the length of each of the n timesteps between now and date T, j corresponds to the number of "up" moves, $n - j$ corresponds to the number of down moves, q corresponds to the risk neutral probability of an up move, $u = e^{\sigma\sqrt{\delta t}}$, $d = e^{-\sigma\sqrt{\delta t}} = \frac{1}{u}$, σ corresponds to annualized volatility, and C_j corresponds to the payoff on the call option after n timesteps and j up moves; i.e., $C_j = Max\left[0, u^j d^{n-j} S - K\right]$. The CRR model is widely considered to be the canonical binomial option pricing model; besides being the best-known and most widely cited binomial model, it also provides a particularly simple matching of volatility with the u and d parameters. Specifically, since $u = e^{\sigma\sqrt{\delta t}}$ and $d = e^{-\sigma\sqrt{\delta t}} = \frac{1}{u}$, it follows that the variance of stock returns is $\sigma^2\delta t$ (see Hull (2015), pp. 286-287). Since $ud = 1.25 \times 0.8 = 1$ in our numerical example, the CRR model implies that $\sigma = \frac{\ln u}{\sqrt{\delta t}} = .2231$.

Here, we recognize that quantitatively challenged students might struggle with standard notation. Thus, we suggest an optional, brief tutorial for using summation notation in the context of this problem. Suggested whiteboard or presentation slide content is provided in Figure 8. Such students might also appreciate a plain-language reading of equation (25). We suggest the following: "The value of a call option is the present value of the weighted average of the values of the call option at expiration, where the weightings represent the risk neutral probabilities of arriving at each terminal node. Thus, today's call option price is simply the present value of this weighted average, discounted at the riskless rate of interest."

Suppose $n = 1$, in which case there is only one timestep and the length of the timestep is $\delta t = T$. Then equation (25) may be rewritten in the following manner:

$$C = e^{-rT}\left[\sum_{j=0}^{1}\begin{pmatrix}1\\ j\end{pmatrix} q^j (1-q)^{1-j} C_j\right] = e^{-rT}\left[(1-q)C_0 + qC_1\right] = e^{-rT}\left[(1-q)C_d + qC_u\right]. \tag{26}$$

In other words, equation (26) is a special case of equation (25), where $n = 1$. Now suppose that n =2. Then,

$$\begin{aligned} C &= e^{-rT}\sum_{j=0}^{2}\begin{pmatrix}2\\ j\end{pmatrix} q^j (1-q)^{2-j} C_j \\ &= e^{-rT}\left[(1-q)^2 C_0 + 2q(1-q)C_1 + q^2 C_2\right] \\ &= e^{-rT}\left[(1-q)^2 C_{dd} + +2q(1-q)C_{ud} + q^2 C_{uu}\right]. \end{aligned} \tag{27}$$

Equations (26) and (27) showcase how the formula given by equation (25) generalizes to the case of n timesteps. In equation (25), $q^j (1-q)^{n-j}$ corresponds to the risk neutral probability of each possible j up move sequence. The $\begin{pmatrix}n\\ j\end{pmatrix}$ term indicates how many j up move sequences exist for each terminal node in the binomial tree, so $\begin{pmatrix}n\\ j\end{pmatrix} q^j (1-q)^{n-j}$ indicates the risk neutral probability of ending up at the j^{th} up move terminal node.

Equation (25) can be simplified even further by rewriting it in such a way that makes it possible to ignore all cases in which the call option is at- or out-of-the-money. Specifically, we need to know the *minimum* number of "up" moves required during the course of n timesteps in order for this to occur. Since the payoff on the call option after n timesteps and j up moves is $C_j = Max\left(0, u^j d^{n-j} S - K\right)$, we need to determine the minimum (non-negative) integer value for j such that the call option will expire in-the-money; i.e., so that $u^j d^{n-j} S > K$. Let b represent the *non-integer* value for j such that the value of the underlying asset would be exactly equal to K at expiration; i.e., $u^b d^{n-b} S = K$. Solving this equation for b,

$$\begin{aligned}
\ln\left(u^b d^{n-b} S\right) &= \ln K \\
b\ln u + (n-b)\ln d &= \ln(K/S); \\
b\ln(u/d) &= \ln(K/Sd^n); \\
b &= \ln(K/Sd^n)/\ln(u/d). \qquad (28)
\end{aligned}$$

Thus, the minimum *integer* value for j such that the call option will expire in-the-money is a, where a is the smallest (non-negative) integer that is greater than b. If $a = 0$, this implies that *all* of the call option payoffs at the end of the tree are positive. If $a = n$, then the only node at which a call option pays off is when there has been n consecutive up moves. In theory, a can exceed n; in that case, the call will always out of the money and therefore worthless.

Since $u^j d^{n-j} S - K > 0$ for all $j \geq a$, equation (25) can be re-written as follows:

$$C = SB_1 - Ke^{-rT}B_2, \qquad (29)$$

where $B_1 = \left[\sum_{j=a}^{n} \begin{pmatrix} n \\ j \end{pmatrix} q^j (1-q)^{n-j} \left(u^j d^{n-j} e^{-rT}\right)\right]$, $B_2 = \left[\sum_{j=a}^{n} \begin{pmatrix} n \\ j \end{pmatrix} q^j (1-q)^{n-j}\right]$, $0 \leq B_1 \leq 1$, and $0 \leq B_2 \leq 1$. Also note that B_1 represents the hedge ratio for the binomial option pricing model and B_2 represents the (risk neutral) binomial probability that the option will expire in-the-money. Furthermore, SB_1 corresponds to today's value of the underlying asset component of the replicating portfolio, whereas $-Ke^{-rT}B_2$ corresponds to today's value of the margin account which is used to partially finance the underlying asset component of the replicating portfolio.

Equation (29) closely resembles the Black-Scholes formula for pricing a European call option. The Black-Scholes formula is written as follows:

$$C = SN(d_1) - Ke^{-rT}N(d_2), \qquad (30)$$

where $d_1 = \dfrac{\ln(S/K) + (r + .5\sigma^2)\,T}{\sigma\sqrt{T}}$, $d_2 = d_1 - \sigma\sqrt{T}, N(d_1)$ and $N(d_2)$ correspond to the standard normal distribution function evaluated at d_1 and d_2 respectively. Like B_1 and B_2, $N(d_1)$ and $N(d_2)$ are bounded from below at 0 and from above at 1. Note that in the "limiting" case (where $T = n\delta t$ remains a fixed value as $n \to \infty$ and $\delta t \to 0$), then B_1 converges in value to $N(d_1)$ and B_2 converges in value to $N(d_2)$. Consequently, the interpretations offered in the previous paragraph for B_1, B_2, SB_1, and $-Ke^{-rT}B_2$ also apply to $N(d_1)$, $N(d_2)$, $SN(d_1)$, and $-Ke^{-rT}N(d_2)$.

The convergence of the Cox-Ross-Rubinstein binomial option pricing formula given by equation (29) and the Black-Scholes option pricing formula given by equation (30) can be shown analytically and numerically. For analytic proofs of how probabilities and prices under the CRR binomial model converge to Black-Scholes probabilities and prices, see Cox, Ross, and Rubinstein (1979) and Hsia (1983). Rendleman and Bartter (1979) independently derive a similar binomial model to that of CRR and provide an analytic proof of the convergence of their model to Black-Scholes in an appendix to their paper. Joshi (2011) also considers various binomial models other than CRR and shows that while the CRR $ud = 1$ assumption is analytically convenient, it is not necessary to obtain convergence to Black-Scholes. In the next section of the paper, we will *numerically* illustrate the convergence of the CRR model to the Black-Scholes option pricing model, and leave analytic illustration for graduate-level courses.

4 Convergence: Numerical

In a spreadsheet model (available at http://bit.ly/options_econ_converge), we numerically illustrate Black-Scholes and CRR model prices based on our executive compensation example in which $S = \$50$, $K = \$60$, $r = 3\%$, $T = 2$ years, $\sigma = .2231$, and the option grant is for 1,000 shares of company stock. Applying the Black-Scholes formula provided in equation (30), we find that $d_1 = \dfrac{\ln(S/K) + (r + .5\sigma^2)\,T}{\sigma\sqrt{T}} = \dfrac{\ln(60/50) + (.03 + .5(.2231^2)\,2}{.2231\sqrt{2}} = -.230$, $d_2 =$

$d_1 - \sigma\sqrt{T} = -.230 - .2231\sqrt{2} = -.545$, $N(d_1) = N(-.230) = .409$, and $N(d_2) = N(-.540) = .293$. Thus, the (Black-Scholes) value of a call option to purchase one share of company stock is \$3.91, which implies that the market value of the option grant is \$3,910.

In Table 1, we list CRR model probabilities and prices (based on equation (29)) alongside the fixed Black-Scholes model probabilities and price (based on equation (30)) obtained from the spreadsheet model. This table shows that as the number of timesteps increases, the frequency at which the call option expires in-the-money at end-of-tree nodes (as indicated by B_2) also varies. Furthermore, the CRR probabilities (as indicated by the B_1 and B_2 columns) and CRR prices swing back and forth as timesteps are added. These swings become less attenuated as the number of timesteps increase, eventually converging toward the Black-Scholes probabilities ($N(d_1) = 0.409$ and $N(d_2) = 0.293$) and \$3.91 price. Figure 9 graphically illustrates the convergence in price and Figure 10 graphically illustrates the convergence in probability. Many of the results obtained from our spreadsheet model (including the "sawtooth" image present in Figure 10) are explained in more complete detail by Feng and Kwan (2012).

5 Conclusion

In this paper, we have provided a simple approach for introducing option pricing models to undergraduate students. We have shown how the delta hedging and replicating portfolio approaches to pricing call and put options imply that risk neutral valuation relationships exist between option prices and the prices of the underlying assets that they reference. After showing the logical connections between these various approaches in a single-period setting, we show how the risk neutral approach generalizes to the multi-period case that is captured by the CRR model. Finally, we numerically and graphically show how in the limit (as $n \to \infty$ and $\delta t \to 0$ for a fixed time to expiration), the prices and probabilities which comprise the CRR pricing equation given by equation (29) converge to the prices and probabilities which comprise the Black-Scholes pricing equation given by equation (30).

To further support instruction of option pricing models, we provide some classroom tools, including a limited managerial compensation case study, whiteboard examples that can help instructors explain and demonstrate the process to their students and a spreadsheet which shows the convergence between the CRR and Black-Scholes models (available at http://bit.ly/options_econ_converge).

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Table 1. Convergence of Cox-Ross-Rubinstein to Black-Scholes

Time Steps	q	B_1	B_2	CRR Value	$N(d_1)$	$N(d_2)$	Black-Scholes Value
1	0.518	0.669	0.518	$4.17	0.409	0.293	$3.91
2	0.512	0.386	0.262	$4.48	0.409	0.293	$3.91
3	0.510	0.215	0.132	$3.29	0.409	0.293	$3.91
4	0.508	0.452	0.325	$4.22	0.409	0.293	$3.91
5	0.507	0.299	0.197	$3.82	0.409	0.293	$3.91
10	0.505	0.517	0.390	$3.83	0.409	0.293	$3.91
50	0.502	0.360	0.250	$3.89	0.409	0.293	$3.91
100	0.502	0.440	0.320	$3.91	0.409	0.293	$3.91
200	0.502	0.387	0.273	$3.91	0.409	0.293	$3.91
500	0.502	0.408	0.307	$3.91	0.409	0.293	$3.91
1000	0.502	0.400	0.285	$3.91	0.409	0.293	$3.91
5000	0.502	0.408	0.292	$3.91	0.409	0.293	$3.91

Note. – Binomial and Black-Scholes values and risk neutral probabilities of an option with the following parameters: S=50, σ=0.2231, u=1.25, d=0.8, t=2, K=60, r =0.03.

Figures

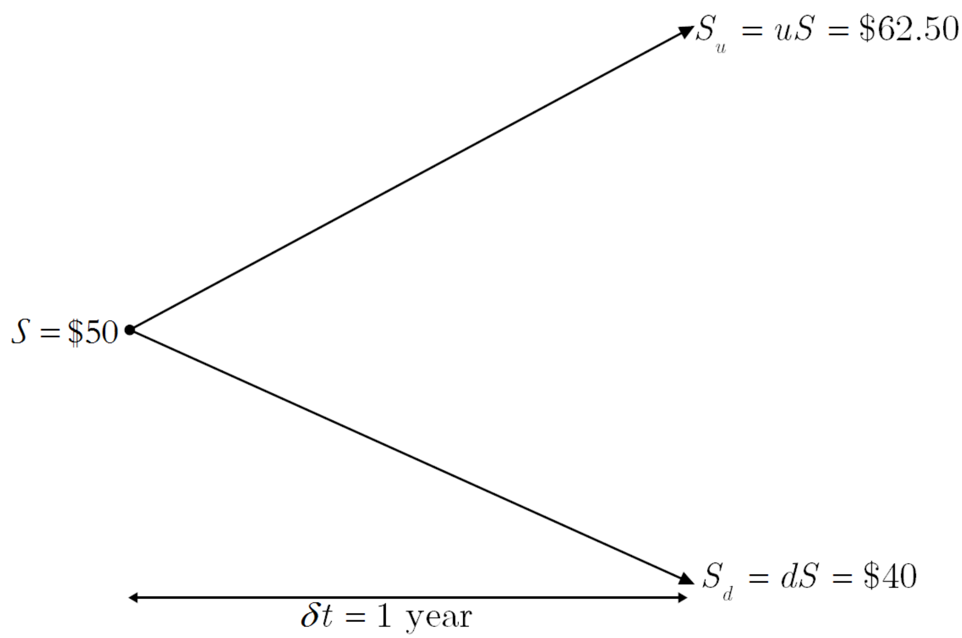

Figure 1. Single-Period Binomial Tree for the Current and Future Stock Prices

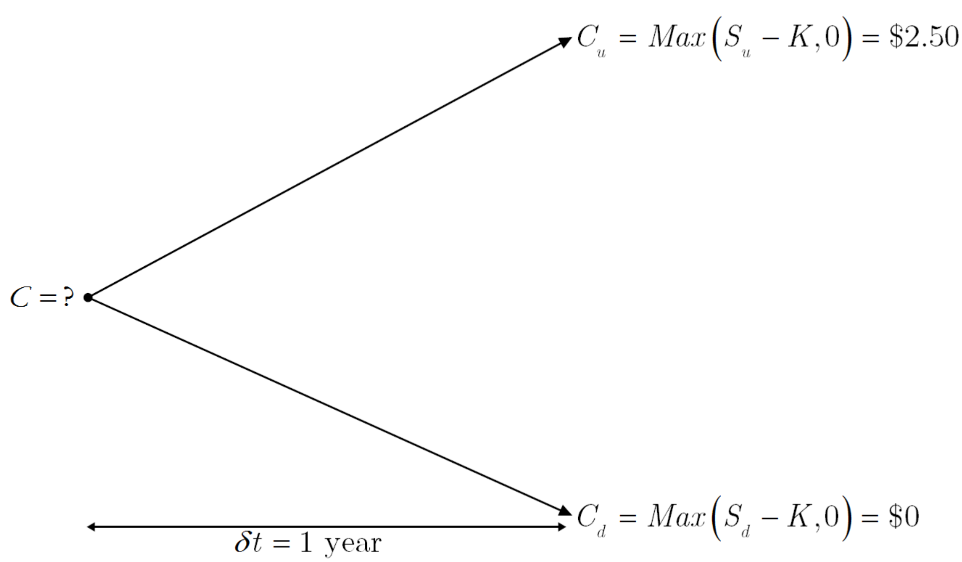

Figure 2. Single-Period Binomial Tree for the Current and Future Call Option Prices

Figures (Continued)

$$C_u - \Delta S_u = C_d - \Delta S_d$$

$$2.50 - \Delta 62.5 = 0 - \Delta 40$$

$$\underline{+\Delta 62.5 \ \ + \Delta 62.5}$$

$$2.50 = \Delta 22.5$$

$$\Delta = 0.111 = \frac{1}{9}$$

Then:

$$V_H^u = 2.5 - (0.111)(62.50)$$

$$= 2.5 - 6.9438$$

$$= -4.44 \ and \ \ldots$$

$$V_H^d = 0 - (0.111)(40)$$

$$= -4.44$$

So:

$$V_H = PV(V_H^u) = PV\left(V_H^d\right) = e^{-0.03}(-4.444) = -4.31$$

Figure 3. Whiteboard Illustration for Finding Hedge Ratio and Present Value of Hedge Portfolio

Figures (Continued)

$$\Delta = \frac{C_u - C_d}{S(u-d)}$$

$$= \frac{2.5 - 0}{50(1.25 - 0.8)}$$

$$= 0.111$$

$$B = \frac{uC_d - dC_u}{e^{r\delta t}(u-d)}$$

$$= \frac{1.25(0) - 0.8(2.5)}{e^{0.03*1*1}(1.25 - 0.8)}$$

$$= \frac{-2}{0.4637}$$

$$= -4.31$$

Figure 4. Whiteboard Illustration for Replicating Portfolio Calculations of Δ and B

$$p = \frac{e^{\mu\delta t} - d}{u - d}$$

$$p(u-d) = e^{\mu\delta t} - d$$

$$pu - pd + d = e^{\mu\delta t}$$

$$pu + (1-p)d = e^{\mu\delta t}$$

$$ln[pu + (1-p)d] = \mu\delta t$$

$$\mu = \frac{ln[pu + (1-p)d]}{dt}$$

Figure 5. Whiteboard Illustration for Deriving Required Return Under Risk (μ)

Figures (Continued)

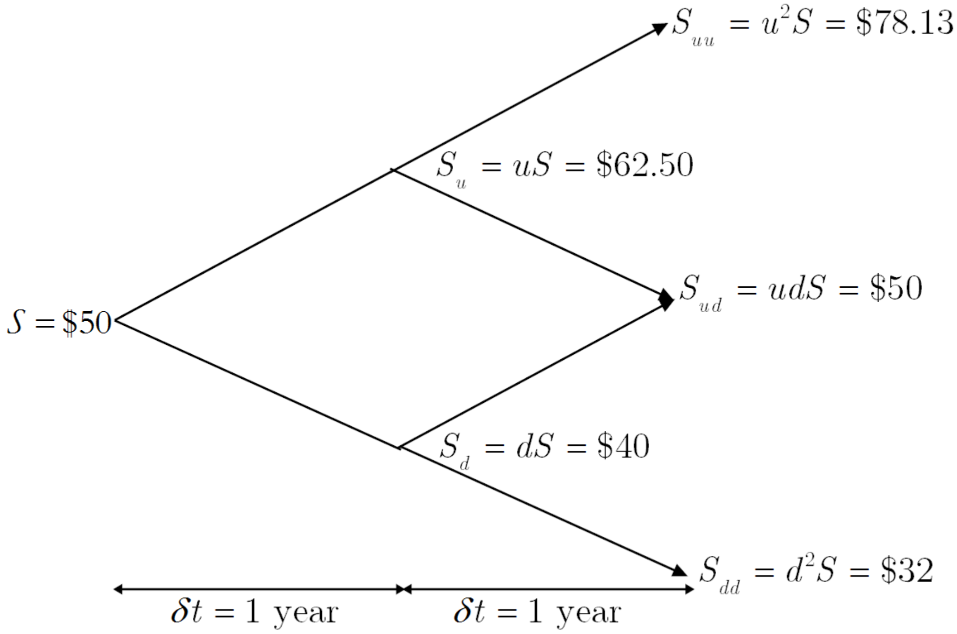

Figure 6. Two-Period Binomial Tree for the Current and Future Stock Prices

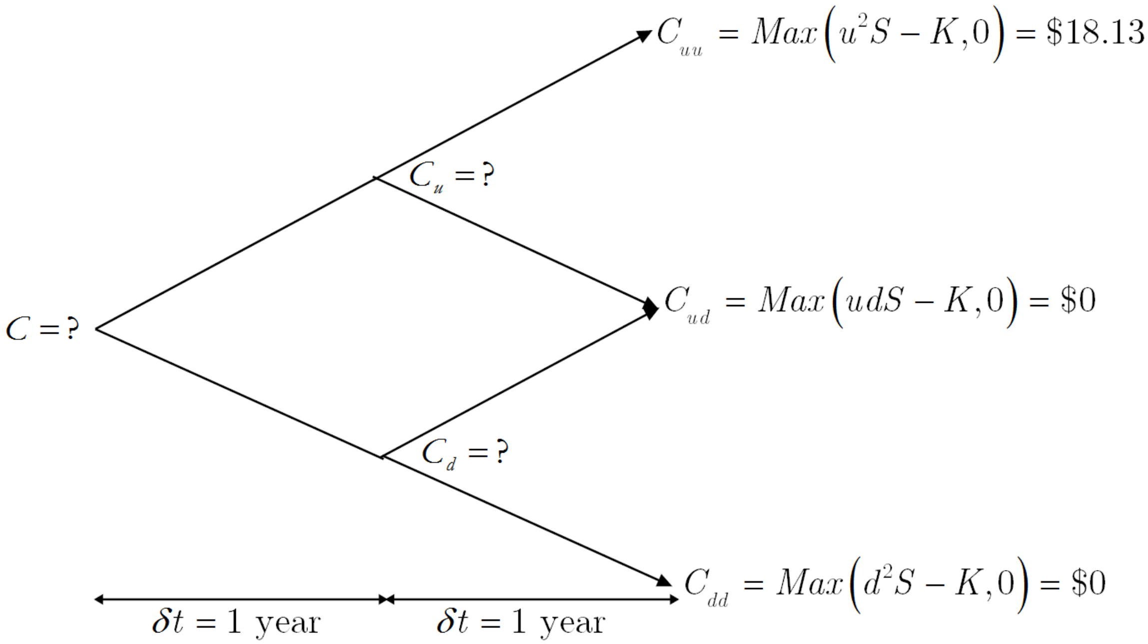

Figure 7. Two-Period Binomial Tree for the Current and Future Call Option Prices

Figures (Continued)

Consider the term in our equation:

$$\sum_{j=0}^{n} \binom{n}{j} q^j (1-q)^{n-j} C_j$$

Where *n*=the number of time steps, *j*=the number of up moves to the terminal node, *q*=risk neutral probability and C_j=the value of the call in terminal node *j*.

The summation symbol tells us to add the simplified expressions for each *j* starting with 0 until the number of time steps (2 in our case). So, we will calculate the expression three times (*j*=0, 1, and 2).

Considering our binomial tree, we know that when *j*=0 (no up moves, ending in node *dd*), the value of C_j is 0, as the option expires out of the money. The result is similar in our case for *j*=1 (one up move, ending in node *ud*, which, in a recombining binomial tree, is also node *du*). That leaves *j*=2 (ending in node *uu*) as the only expression for which we need to simplify the expression.

First, we calculate $\binom{n}{j}$, which is notation for $\frac{n!}{j!(n-j)!} = \frac{2!}{2!(2-2)!} = 1$. Then, we substitute *q*, *n*, *j*, and C_j into the expression:

$$(1)(0.5121^2)(0.4879^{2-2})(18.13) = 4.75$$

Now, we add the three values of this expression: $0 + 0 + 4.75 = 4.75$ and continue solving the equation.

Figure 8. Whiteboard Example: Explanation of Equation (25) Summation Notation

Figures (Continued)

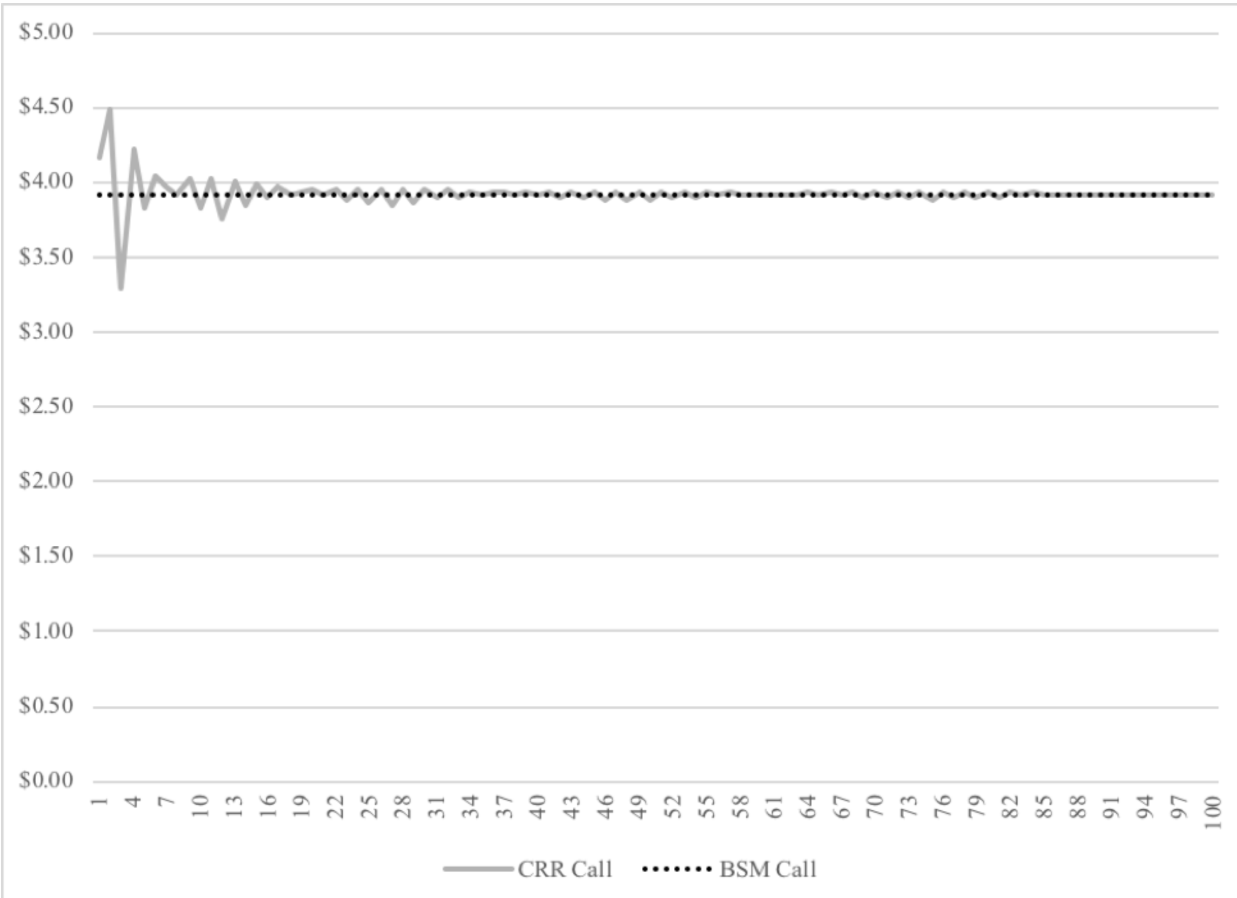

Figure 9. Convergence of Cox-Ross-Rubinstein (CRR) to Black-Scholes Model (BSM) Prices

Note. – Binomial and Black-Scholes values of an option with the following parameters: S=50, σ=0.2231, u=1.25, d=0.8, t=2, K=60, r =0.03. Number of time-steps represented on the x-axis.

Figures (Continued)

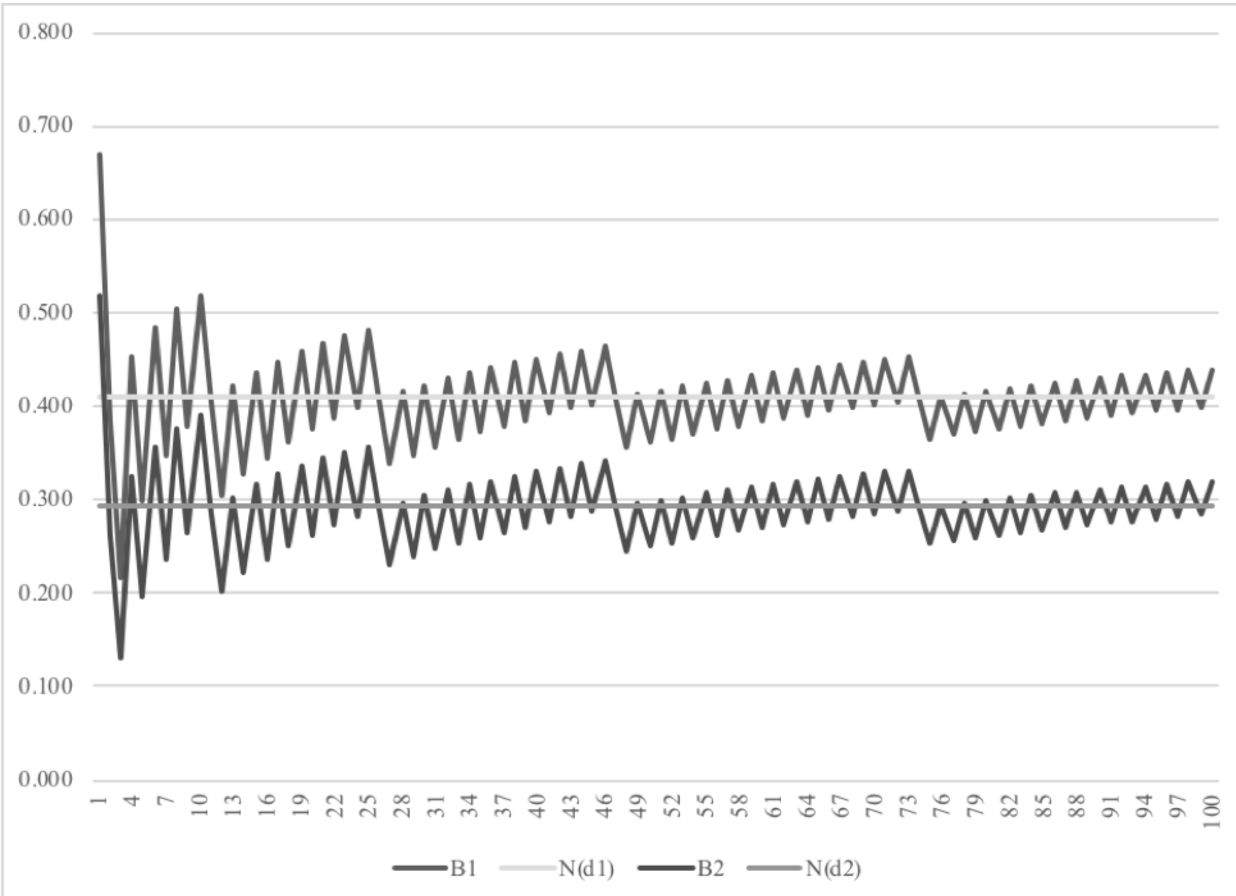

Figure 10. Convergence of Cox-Ross-Rubinstein (CRR) to Black-Scholes Model (BSM) Probabilities

Note. – Binomial and Black-Scholes risk neutral probabilities of an option with the following parameters: S=50, σ=0.2231, u=1.25, d=0.8, t=2, K=60, r =0.03. Number of time-steps represented on the x-axis.